# Further developments in the Discontinuous Galerkin based dynamical core for ICON (BRIDGE)

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# The BRIDGE project (Basic Research for ICON with DG Extension)

**BRIDGE** is an *informal* project *at DWD*, started ~ mid 2020

currently: *F. Prill, M. Baldauf, D. Reinert*

**Goals**:

- develop a prototype for a **Discontinuous Galerkin (DG)** discretization of the 3D Euler equations (‚DG-HEVI on the sphere')
- together with a minimal set of physical parameterizations
- using ICON infrastructure (parallelisation via YAXT, I/O, ...)
- more object-orientation and use of standard software (e.g. YAC coupler, ...)

→ **BRIDGE** is an intermediate step to a full–fledged ICON implementation.

Later on (starting ~2024), the BRIDGE code will serve as a code base for the final **ICON-DG model**.

# Discontinuous Galerkin (DG) methods in a nutshell

$$\frac{\partial q^{(k)}}{\partial t} + \nabla \cdot \mathbf{f}^{(k)}(q) = S^{(k)}(q), \qquad k = 1, ..., K$$

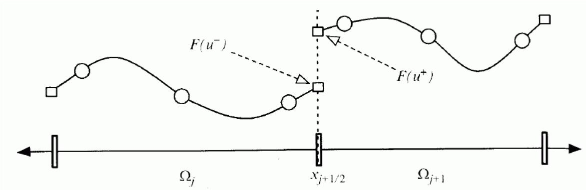
e.g.
*Cockburn, Shu (1989) Math. Comput.*
*Cockburn et al. (1989) JCP*
*Hesthaven, Warburton (2008)*

1.) weak formulation

$$\frac{d}{dt} \int_{\Omega_j} q^{(k)} \, v \, dV + \int_{\partial \Omega_j} f^{(k) num, \perp} \, v \, da - \int_{\Omega_j} \mathbf{f}^{(k)} \cdot \nabla v \, dV = \int_{\Omega_j} S^{(k)} \, v \, dV$$

2.) Finite-element ingredient

$$q^{(k)}(x, t) = \sum_{l=0}^{p} q_{j,l}^{(k)}(t) \; p_l(x - x_j)$$

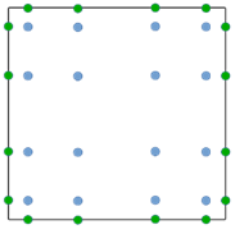



From *Nair et al. (2011)* in *‚Numerical techniques for global atm. models'*

3.) Finite-volume ingredient: numerical flux

$$\mathbf{f}(q) \to f^{num,\perp}(q^+, q^-) = \frac{1}{2} \left( \mathbf{f}(q^+) + \mathbf{f}(q^-) \right) \cdot \mathbf{n} - \frac{\alpha}{2}(q^+ - q^-)$$

4.) Gaussian quadrature for the volume and surface integrals



→ ODE-system for $q^{(k)}{}_{jl}(t)$

5.) Use a time-integration scheme (Runge-Kutta, …)

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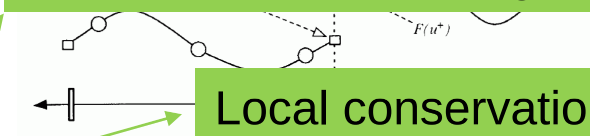
higher order discretization ✓

2.) Finite-element ingredient

works on unstructured grids ✓

$$q^{(k)}(x,t) = \sum_{l=0}^{p} q^{(k)}_{j,l}(t)\; p_l(x - x_j)$$



Local conservation ✓

From *Nair et al. (2011)* in ‚*Numerical techniques for global atm. models*'

3.) Finite-volume ingredient: numerical flux

$$\mathbf{f}(q) \to f^{num,\perp}(q^+, q^-) = \frac{1}{2}\left(\mathbf{f}(q^+) + \mathbf{f}(q^-)\right) \cdot \mathbf{n} - \ldots$$

allows explicit time integration ✓

4.) Gaussian quadrature for the volume and surface integrals

→ ODE-system for $q^{(k)}{}_{jl}(t)$

allows massive parallelisation ✓

5.) Use a time-integration scheme (Runge-Kutta, ...)

high computational intensity ✓

# Improvements in the  efficiency of the HEVI solver

# Efficiency of the HEVI (vertically implicit) solver

The HEVI approach applied to DG leads to (large) *block-tridiagonal linear system of equations* (*Baldauf (2021) JCP*).

Problem: the complexity of a direct solver for a linear system of $N$ equations for LU-decomposition is $O(N^3)$, for matrix vector mult. is $O(N^2)$ with
$N$ = # of variables (5) * # of base functions in each cell (10*4) (for 4th order)

Efficiency improvement by
**Collocation** (i.e. interpolation points = quadrature points)
This decouples the equations horizontally (also in the BCs!) →
$N$ = # of variables (5) * # of vertical base functions in each cell (4)

→ **HEVI solver ~ 10 times faster than before**!

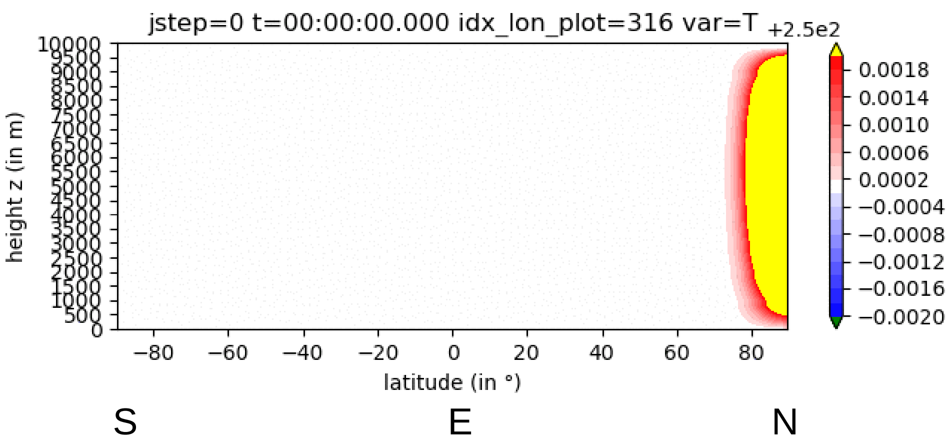
→ we are in the game again ☺

## 3D Euler equations:
## quasi-linear expansion of gravity and sound waves in a spherical shell

*Baldauf, Reinert, Zängl (2014) QJRMS* derive a linearized analytic solution for

- test scenario (A): $f = 0$
- test scenario (B): $f = 10 * f_{geo}(45°)$ → $f / N \sim 0.05$

DG 4th order scheme with **explicit** time integration (4th order Runge-Kutta)

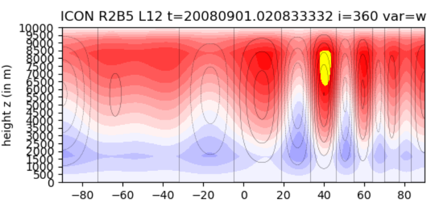



‚small planet‘:
$R = R_{earth} / 50$

# A very first (!), almost (!) fair comparison between ICON and BRIDGE

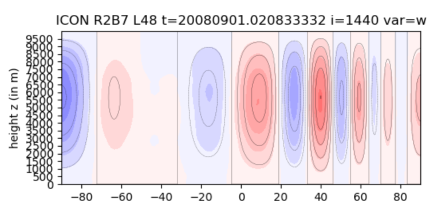
- test case *Baldauf, Reinert, Zängl (2014) QJRMS*: expansion of sound- and gravity waves in a spherical shell (uses shallow atmosph. approx., $U_0=0$, $f=0$). **Analytic solution** available!
- $R = r_{\text{earth}} / 100$ → almost **isotropic grid cells**
  → HEVI- (ICON) and purely expl. time integration (BRIDGE) are comparable!
- all simul. on our RCL (ICON: 32, 64, 384 proc. (thanks to D. Reinert!), BRIDGE: 16 proc.)

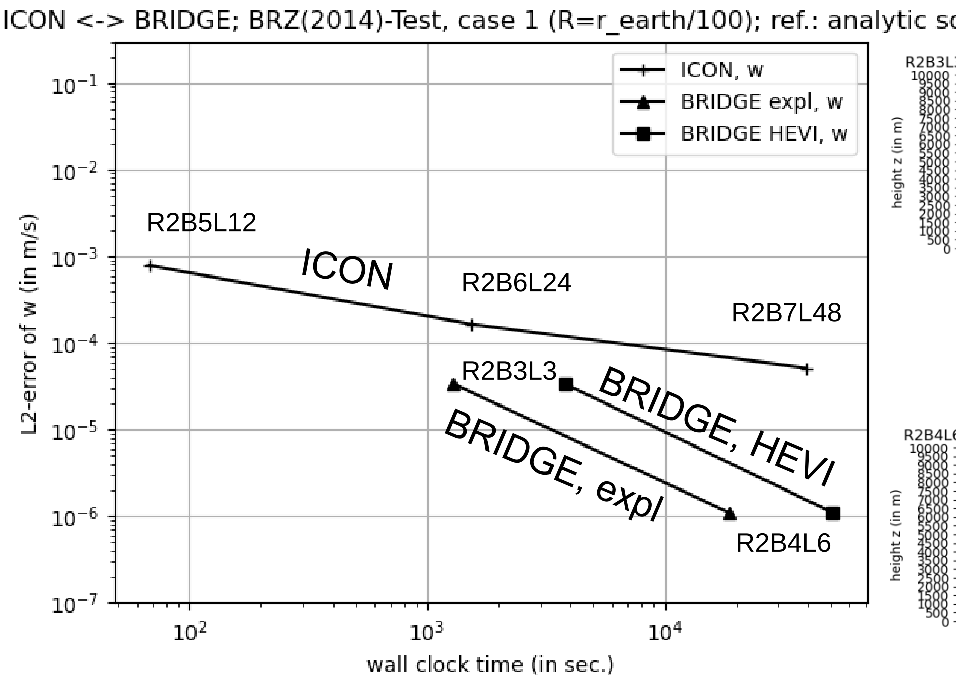
R2B5L12



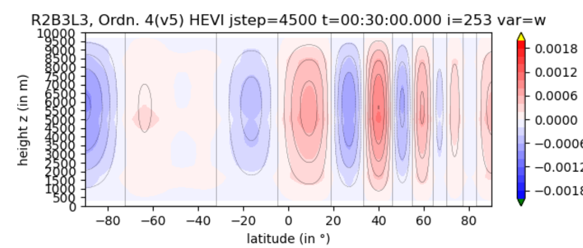

ICON

R2B7L48



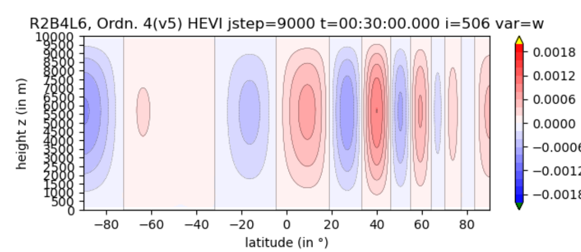




R2B3L3




BRIDGE

R2B4L6



## Choice of prognostic variables

## **Euler equations** (in covariant form)

Continuity eq.: $\frac{\partial \rho}{\partial t} + \nabla_k M^k = 0$ $\quad$ $M^k = \rho v^k$

Momentum eq.: $\frac{\partial M^i}{\partial t} + \nabla_k T^{ik} = S^i_{(M)}$

$$T^{ik} = \frac{1}{\rho} M^i M^k + p g^{ik} \qquad S^i_{(M)} = -g \rho e^i - 2 g^{il} \epsilon_{ljk} \Omega^j M^k$$

Energy budget eq.:

Variant 1: use density weighted potential temperature $\theta = \rho \Theta$

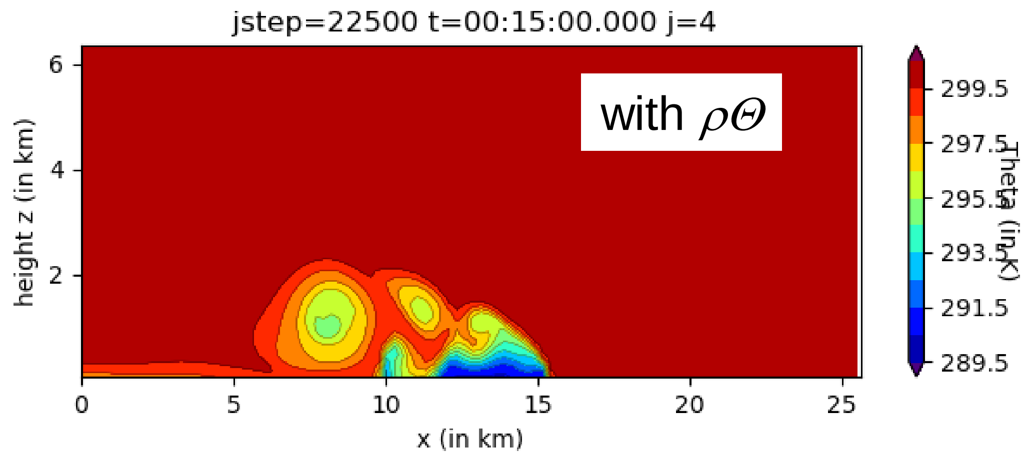
$$\frac{\partial \theta}{\partial t} + \nabla_k \, v^k \theta = 0 \qquad p = p(\theta)$$

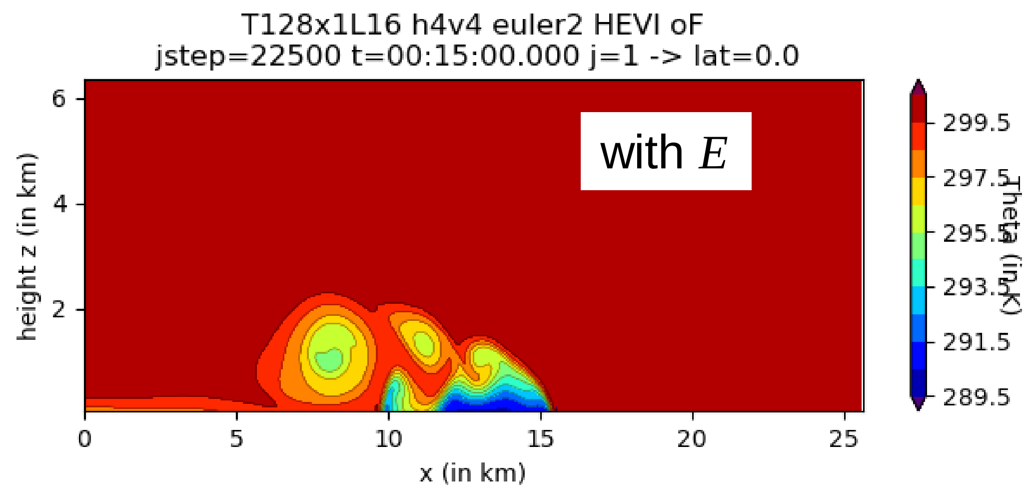
Variant 2: use total energy density $E = E_{kin} + E_{pot} + E_{int}$

$$\frac{\partial E}{\partial t} + \nabla_k \, v^k (E + p) = 0 \qquad p = p(E_{int})$$

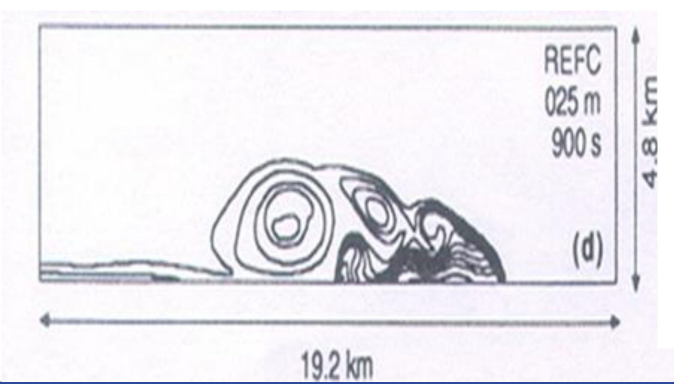
## Falling cold bubble in a viscous medium

BRIDGE, 4$^{th}$ order, RK4, $\Delta x = \Delta z = 400$m




Reference solution
for $\theta$ from
*Straka et al. (1993)*
($\Delta x = \Delta z = 25$m)

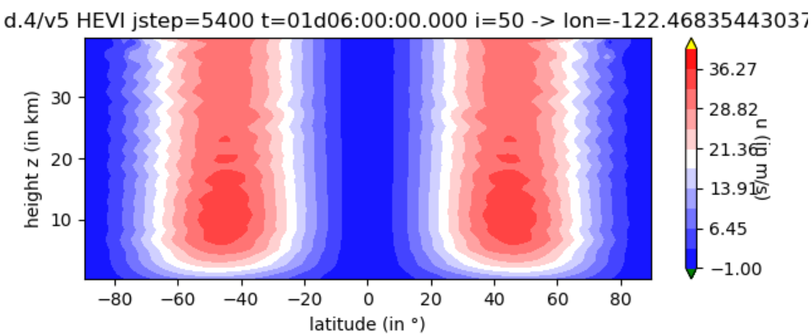


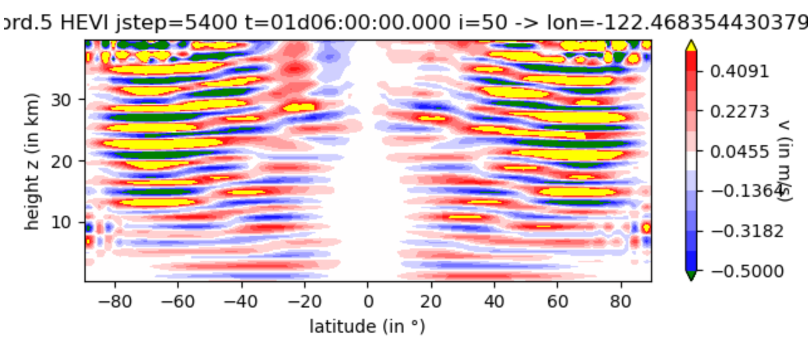
# Baroclinic instability test on the sphere

*Jablonowski, Williamson (2006) QJRMS*

R2B2L10, 4th order, after t=30h

## With $\rho\theta$, source term filtering

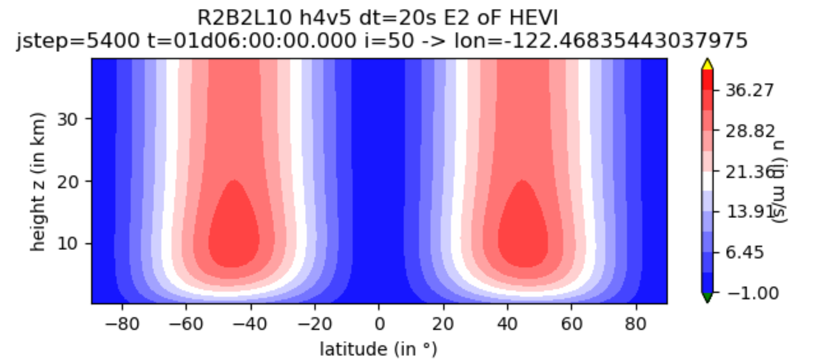


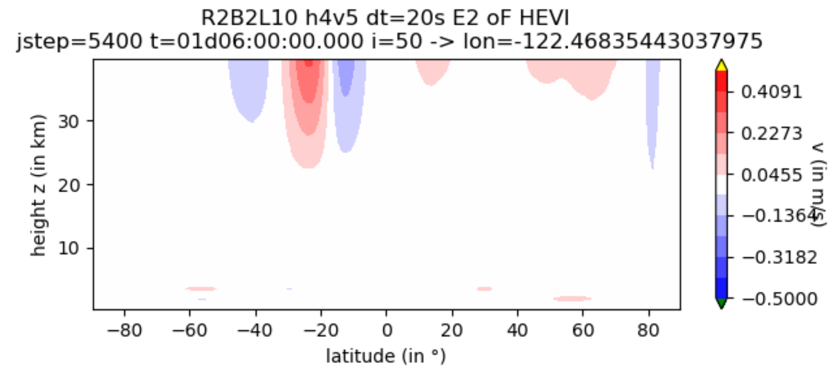




**Becomes quickly unstable!**

Observation: slowly increasing disturbances in the vertical.

## With $E$, no src term filtering




Clear advantage by the use of $E$

## Baroclinic instability test on the sphere

*Jablonowski, Williamson (2006) QJRMS*

- With $\rho\Theta$, the simulation breaks after only ~30h, obviously due to heavy well-balancing problems
- **Using total energy** $E$ and without any further actions, the test runs about 7d and then breaks
- Tests with any kind of physical diffusion (Smagorinsky or K=const.) either damps the unstable wave too much or doesn't prevent instability
- **Exponential filtering** (e.g. *Hesthaven, Warburton (2008)*):
  in each time step reduce a bit the amplitude in dependence of the polynomial degree (but no change of degree 0! ← conservation)
  This is a relatively cheap proxy for hyperdiffusion;
  but acts separately on each grid cell
- With the filter settings for exponential filtering found until now, the test runs until 15d
- → Obviously, further testing is necessary
- Far future: use kinetic energy conserving DG schemes (*Gassner et al. (2021)*, …)

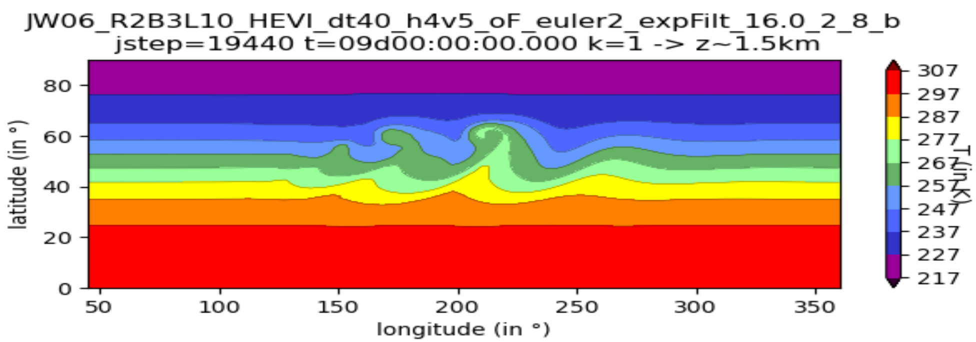
# Baroclinic instability test on the sphere

*Jablonowski, Williamson (2006) QJRMS*

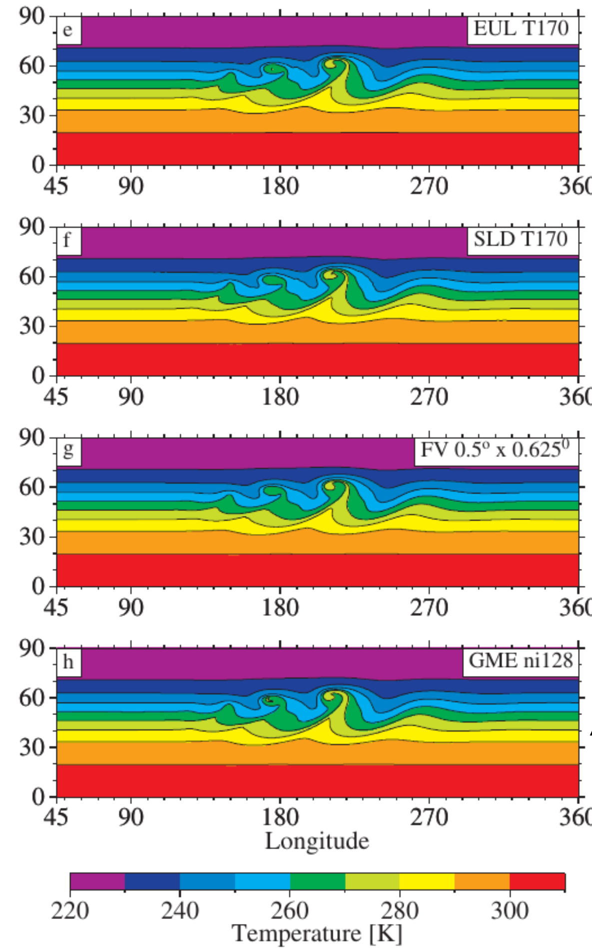
BRIDGE DG 4th order

R2B3 → $\Delta x$ ~ 315 km, $\Delta t$=40s

T in z=1.5km




Feld 1: Min=224.6468132269569, Max=301.052790588424




$\Delta x$ ~ 60 km

Stabilization by an exponential filter (formally of 8th order)

$T$ in 850 hPa at $t$=9d (Fig. 7 from *JW06*)

## The DG scheme applied to the TKE equation

## Example: Turbulence closure by a one-equation TKE model

Density of the turbulent kinetic energy (TKE density): $\bar{e} := \frac{1}{2} \bar{\rho}\, \overline{v'^i v'_i}$

TKE equation: $\frac{\partial \bar{e}}{\partial t} = -\nabla_k (\bar{v}^k \bar{e}) + \nabla_k (g^{kj} K_e \nabla_j \bar{e}) - \bar{\rho}\, \overline{v'^i v'^j}\, \nabla_j v^k g_{ik} + g \frac{\bar{\rho}}{\bar{\Theta}} \overline{v'^i \Theta'} g^3_i - \epsilon$

Parameterizations:

$$\begin{aligned} \overline{v'^i v'^j} &= -K_M 2 D^{ij} + \frac{2}{3} g^{ij} \bar{e} \\ \overline{v'^i \Theta'} &= -K_H g^{ij} \frac{\partial \bar{\Theta}}{\partial x^j} \\ \epsilon &= C_\epsilon \bar{\rho} \frac{1}{l} \left( \frac{\bar{e}}{\bar{\rho}} \right)^{3/2} \end{aligned}$$

Diffusion coefficients:

$$K_x = c_x l_m \sqrt{\bar{e}}, \quad x = M, H, e$$

Mixing lengths (here only neutral case):

$$l_m = \frac{\kappa z}{1 + \frac{\kappa z}{l_B}}$$

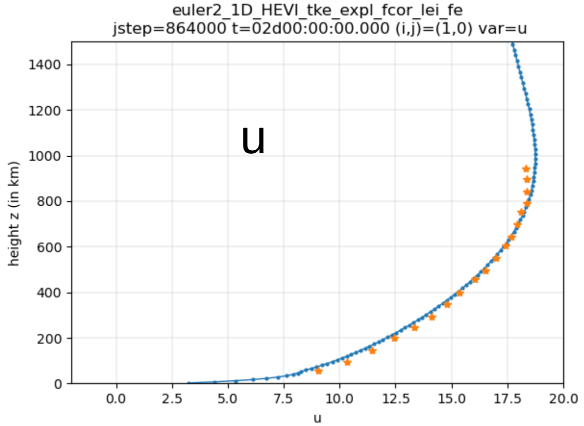
BCs from a simplified ‚transfer scheme' (wall law) for neutral stratification:

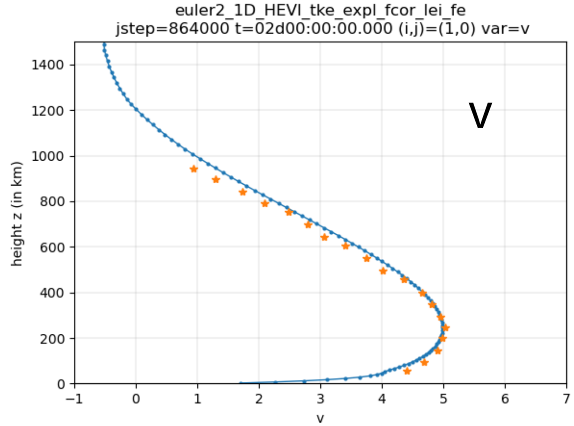
u* from $u(z) = \frac{u_*}{\kappa} \log \frac{z}{z_0}$ → in $z$=0: $\overline{u'w'} = u_*^2$, $\mathbf{v}=0$ and $\bar{e} = c \rho\, u_*^2$

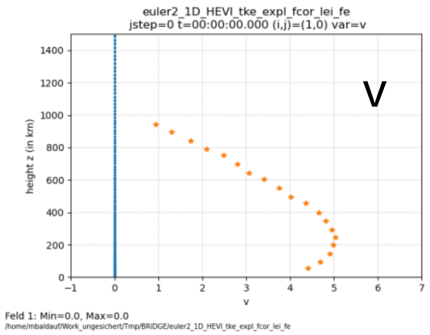
Note: it is not the purpose of this implementation to have a ‚good' turbulence parameterization, but to have the main ingredients that must be considered for a DG treatment!

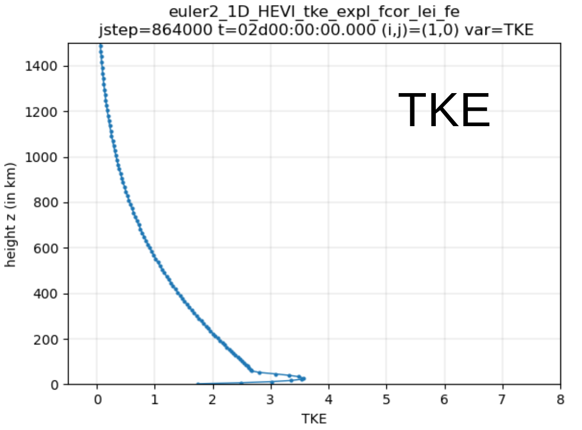
# Euler equations + TKE-turbulence model

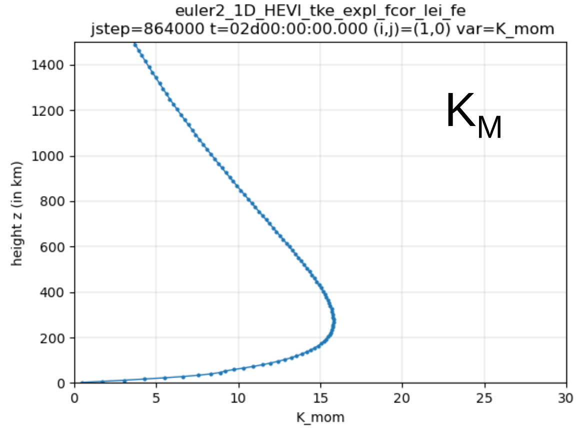
Simulation of the ‚Leipzig wind profile' (from 1931!) $u_g$=17.5 m/s, $z_0$=0.3m, $l_B$=40m
(u, v obs. data from *Detering, Etling (1985)*)



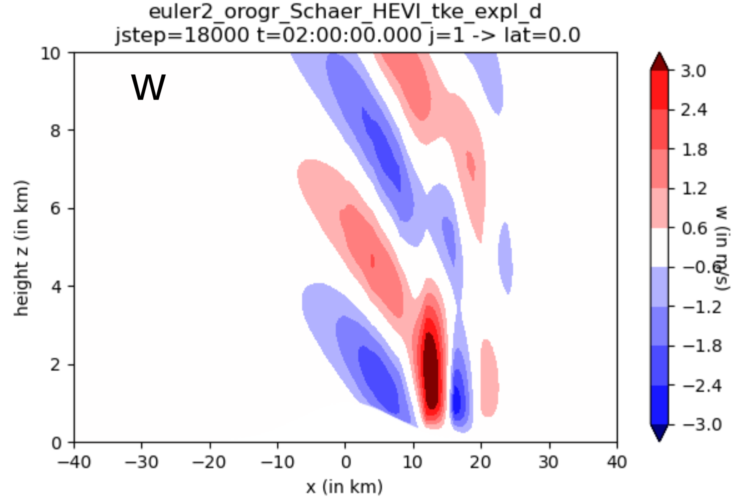



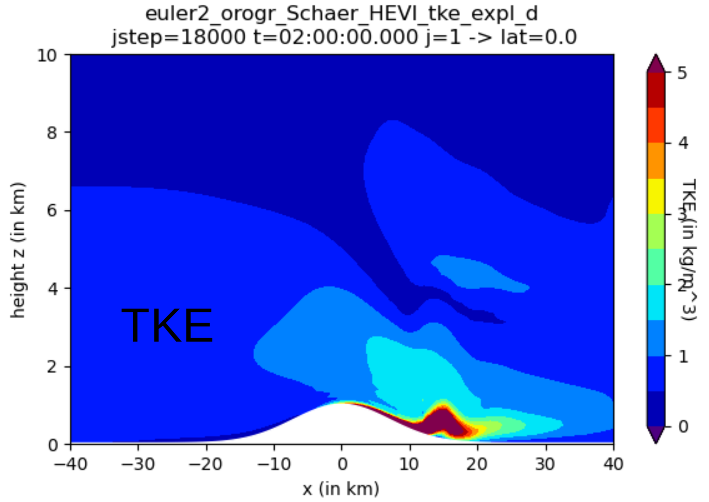

Observed:
u* ~ 0.65 m/s

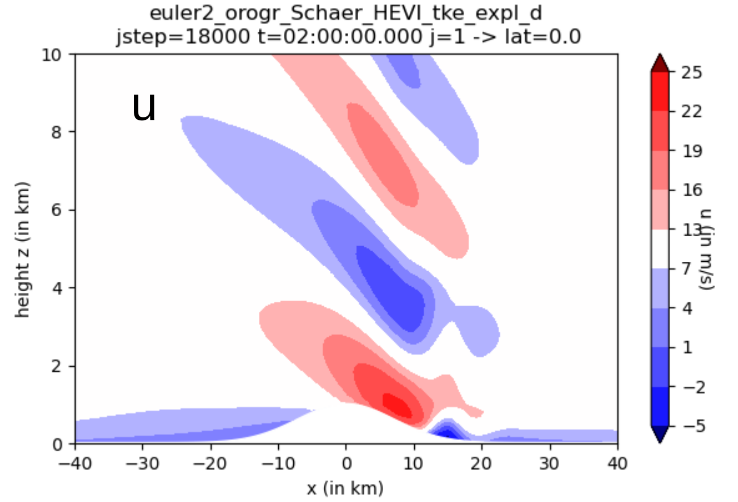
Simulated:
u* ~ 0.667m/s

# Euler equations + TKE turbulence model:

Flow over mountains
$u_0$=10m/s, $N$=const
$\Delta x$=4 km
$z_0$=0.3m



# Milestones of the BRIDGE-project (status Sept. 2023)

| | | |
|---|---|---|
| 0. | first version of a **FE/DG framework** available, **MPI** parallelized | Q2/2021 ✓ |
| 1. | **shallow water** equations on the sphere, **explicit** time integration (RK) | Q3/2021 ✓ |
| 2. | **3D Euler** equations on the sphere, **explicit** time integration (RK) | Q1/2022 ✓ |
| 2.b | with 3D diffusion (+ a simple turbulence scheme) BR1/BR2 | Q2/2022 ✓/🛠 |
| 2.c | grid refinement works | Q2/2022 ✓ |
| 3. | **Euler** equations, **HEVI** time integration (IMEX-RK) | Q3/2022 ✓ |
| 3.b | optimization of the vertically implicit solver (Schur compl.,...) | Q4/2022 ✓/🛠 |
| 3.c | with 3D diffusion (+ a simple turbulence scheme), HEVI | Q1/2023 ✓ |
| 4. | cloud **microphysics** (Kessler) + **tracer advection** (positive definit) + explicit sedimentation scheme | Q4/2022 🛠 ! |
| 4.b | cloud microphysics (cloud ice, Graupel) + vertically implicit sedimentation scheme | Q1/2023 |
| 4.c | HEVI diffusion on physics time step; overall time integration | Q4/2023 |
| 5. | **limited area** version available | Q1/2023 |
| 5.b | **vectorized** version available | Q2/2023 |
| 5.c | **GPU** version | Q1/2024 |
| 6. | coupling of a full fledged **turbulence + BL** scheme | Q2/2023 🛠 |

## Summary(I): BRIDGE, current status (Sept. 2023)

- DG-solver with **explicit time integration** (RK schemes)
  or **implicit-explicit time integration** (IMEX-RK schemes) for **HEVI**
- **Coordinate systems**: on the sphere, ellipsoid or flat plane (3D or x-z) and terrain-following in case of orography
- **Nodal** base
- **Tensor product** representation on prism grid cells
- **MPI** parallelisation, using YAXT library by DKRZ DEUTSCHES KLIMARECHENZENTRUM
- **Grid refinement** on non-conformal grid
- Shallow-water or 3D **Euler equations** (compressible, non-hydrostatic, deep atmosph. or shallow atmosph. approx.)
- Simplified **turbulence** parameterizations (Prandtl, TKE eq., Smagorinsky)
- **Diffusion** via Bassi, Rebay 1 (expl. or HEVI) (BR2 under devel.)

Currently, `bridge/src/` contains ~80000 lines of code

## Summary (II)

- Large efficiency improvement in the **HEVI solver by collocation**
  → horizontal decoupling of FE amplitudes in one grid column!
  Further efficiency improvements by Schur complement technique?
- Equation system using **total energy** $E$ has clear advantages with regard to a **well-balancing problem** in comparison to the use of $\rho\Theta$
  - Source term filtering is no longer needed (since no problems to cure with the buoyancy term)
  - No other detrimental effects visible by using $E$
- The **baroclinic instability test case** (JW06) is quite demanding and needs totel energy $E$ formulation.
  High order exponential filtering stabilizes this case.
- Successful treatment of a **one-equation TKE turbulence model** with the DG approach could be demonstrated.
- Now, real **orography data** can be used, work has been started….

## Summary (III)

- Coupling of a **turbulence model**
  - The wish for local conservation requires a **separation of the turbulence model** into a transport part (DG treatment) and a ‚pure' parameterization part
  - Use of **total energy** as prognostic variable is quite probable
    → instead of developing a new turbulence scheme, can we just express the **new covariances by known** ones (+ only a few additional covariances)?
  - Extend all considerations for **moisture** and by moist transport equations!
  - Similar considerations are in principle necessary for convection param., gravity wave param., …
- Many problems still must be solved in BRIDGE:
  - Collocation for use of different bases
  - Positive definite and mass consistent tracer transport; microphysics coupl.
  - Consistent higher (3rd) order time integration with multiple time steps
  - Overall efficiency
  - …
- Nevertheless, we hope to start the marriage of **BRIDGE** with **ICON** in 2024 …