A semi-implicit non-hydrostatic dynamical kernel using spectral representation in the horizontal and finite elements in the vertical.

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Euler equations for the dry air case (vertical slice)

- The Euler equations are

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} + R\, e^{r}\, \nabla v + \nabla \phi &= \mathbf{F} \\
\frac{d r}{dt} + \frac{R}{C_v} \left(\nabla \cdot \mathbf{v}\right) &= \frac{Q}{C_v\, e^{r}} \\
\frac{d q}{dt} + \frac{C_p}{C_v} \left(\nabla \cdot \mathbf{v}\right) &= \frac{Q}{C_v\, e^{r}}
\end{aligned}
$$

- Prognostic variables are $q = \ln p$, $r = \ln T$ and $\mathbf{v} = (u, w)$

T is the temperature, p the pressure $\mathbf{v}$ the velocity vector, R is the gas constant for dry air, C_p the specific heat capacity of dry air at constant pressure, C_v the specific heat capacity of dry air at constant volume, $\mathbf{F}(t, x, z)$ is the diabatic momentum forcing, $Q(t, x, z)$ the heat per unit mass and unit time added to the air, $\phi(z) = gz$ the geopotential, $\nabla \phi$ the gradient of geopotential, ∇q the gradient of the logarithm of pressure and $\nabla \cdot \mathbf{v}$ the divergence of the velocity

Height based vertical coordinate

- The original Cartesian (x, y) coordinates are transformed into model coordinates (X, Z) such that bottom and top boundaries are $Z = 0$ and $Z = 1$ respectively
- The spatial domain in Cartesian coordinates is bounded by a rigid top at $z = H_T$ and a rigid bottom at $z = H_B(x)$
- The coordinate transformation is written as

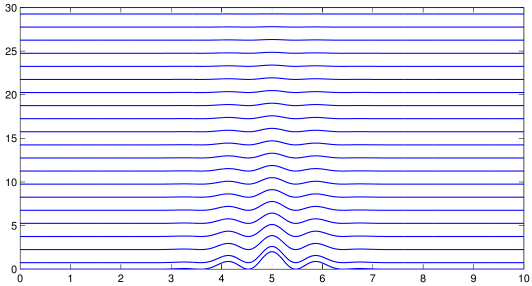
$$
\begin{aligned}
x(X, Z) &= X \\
z(X, Z) &= \psi(X, Z)
\end{aligned}
$$

where $\psi(X, Z)$ satisfies the boundary conditions

Height based vertical coordinate

- One parameter height hybrid coordinate (Schär, 2002)

$$\psi(X,Z) = H_T(1-Z) + H_B(X)\,\frac{\sinh(\gamma Z)}{\sinh(\gamma)}$$



20 level model with Z levels regularly spaced

Vertical discretization

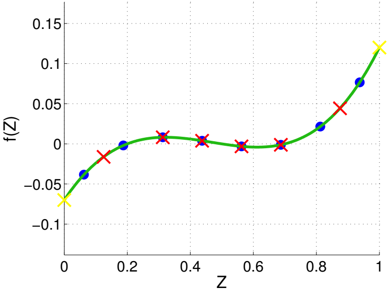
- Similarly to the IFS VFE scheme (Untch and Hortal, 2004) there are not staggered variables
- Model levels are surfaces of constant vertical coordinate ($0 < Z_i < 1$ for $1 \leq i \leq N$) and boundary conditions are applied at $Z_0 = 0$ and $Z_{N+1} = 1$
- Splines in the closed interval $Z \in [0, 1]$ are used for interpolation (within the definition of the discrete vertical operators)
- A B-spline set of basis functions is constructed following the recursive de Boor Algorithm (de Boor, 2001)

Vertical discretization

- There are three types of boundary conditions
  - value of the function is zero (for the contravariant vertical velocity)
  - value of the derivative of the function is zero (for second derivative operators used in diffusion terms)
  - functions without boundary condition functions (for instance logarithm of pressure).
- A set of discrete vertical derivative operators are constructed for each type of boundary conditions

Cubic B-splines interpolation

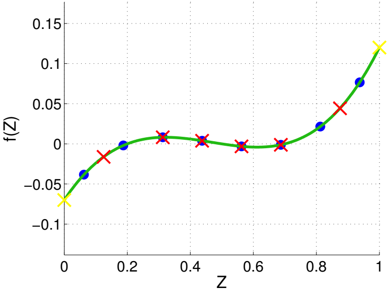
- First step is to interpolate from model levels to spline space



- There are N model levels and $N - 3 + 1$ breakpoints
- There are 2 boundary conditions
- Piecewise polynomials $\phi(Z)$, $\phi'(Z)$ and $\phi''(Z)$ are continuous

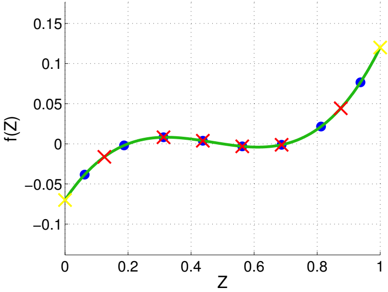
Cubic B-splines interpolation

- Different interpolations depending on the boundary conditions



- Different interpolations depending on the boundary conditions
- For the vertical velocity $\phi(0) = \phi(1) = 0$
- If there are not boundary conditions then $\phi'''(0) = \phi'''(1) = 0$

Cubic B-splines interpolation

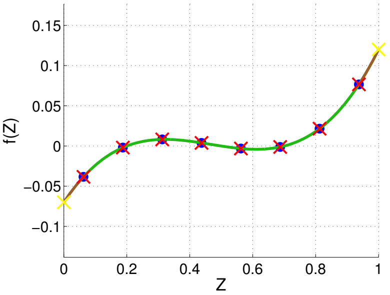


- The number of levels is equal to the number of degree of freedom in the determination of the polynomial coefficients: $N = N_P - N_C - N_B$
- Model levels: N
- Polynomial coefficients: $N_P = (N - 3 + 2) \cdot (3 + 1)$
- Continuity conditions: $N_C = (N - 3 + 1) \cdot 3$
- Boundary conditions: $N_B = 2$

Alternative B-splines interpolation

- All model levels are breakpoints
- Suitable for splines of any degree, tested for 1, 3 and 5

- Extrapolation from first and last levels to boundaries with lower order splines
  $L < K$
- Polynomial coefficients:
  $N_P = (N-1) \cdot (K+1) + 2 \cdot (L+1)$
- Continuity conditions:
  $N_C = N \cdot K$
- Boundary conditions:
  $N_B = 2$
- The number of levels must be equal to the degree of freedom in the determination of the polynomial coefficients:
  $N = N_P - N_C - N_B \Rightarrow L = \frac{K+1}{2}$



Garlekin Method in spline space

- Following (Untch and Hortal, 2004) the Garlekin method is applied in the spline space.
- Given a continuous vertical operator $\mathcal{D}$ the discrete vertical operator is represented by the matrix

$$\mathbf{D} = \mathbf{B}\,\mathbf{M}^{-1}\,\mathbf{N}\,\mathbf{A}$$

where **A** is the projection matrix onto finite element space, **B** is the projection matrix from the spline space to the model levels and the matrices **M** and **N** are

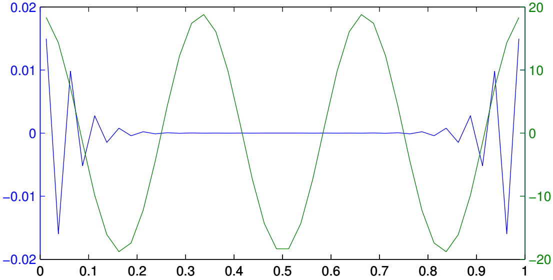
$$M_{ij} = \int_0^1 \mathcal{B}_i(Z)\,\mathcal{B}_j(Z)\,dZ$$

$$N_{ij} = \int_0^1 \mathcal{B}_i(Z)\,\mathcal{D}\left(\mathcal{B}_j(Z)\right)\,dZ$$

where $\mathcal{B}_i(Z)$ are the elements of the B-spline basis

Test: first derivative

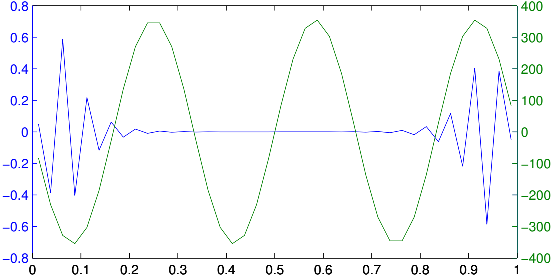
- The numerical error of the discrete vertical derivative for the test function $f(Z) = \sin(6\pi Z)$ is plotted in the figure.



- The number of levels is 40, from 0.0125 to 0.9875 at a constant interval of $\Delta Z = 0.0250$.

Test: second derivative

- The numerical error of the discrete vertical second derivative for the test function $f(Z) = \sin(6\pi Z)$ is plotted in the figure.



- The number of levels is 40, from 0.0125 to 0.9875 at a constant interval of $\Delta Z = 0.0250$.

Covariant formulation: metric tensor

- The formulation of the model only uses "covariant objects" like "covariant derivative", "covariant and contravariant vectors", "natural coordinate basis"
- Following the Riemannian Geometry theory, the first step is to find the expression of the metric tensor in the new coordinates with the help of the Jacobian of the coordinate transformation

$$G_{XZ} = \begin{pmatrix} 1 & 0 \\ \psi_X & \psi_Z \end{pmatrix}^T G_{xz} \begin{pmatrix} 1 & 0 \\ \psi_X & \psi_Z \end{pmatrix} = \begin{pmatrix} 1+\psi_X^2 & \psi_X\psi_Z \\ \psi_X\psi_Z & \psi_Z^2 \end{pmatrix}$$

where ψ_X and ψ_Z are derivatives of $\psi(X,Z)$ and G_{xz} is the metric in Cartesian coordinates which is the identity

Covariant formulation: differential operators

- Differential operators are calculated from the metric tensor and its inverse (G_{ij}, and G^{ij}) and the Christoffel symbols Γ^i_{jk}

$$\Gamma^i_{jk} = \frac{1}{2} G^{im} \left(\frac{G_{mj}}{\partial X^k} + \frac{G_{mk}}{\partial X^j} - \frac{G_{jk}}{\partial X^m} \right)$$

- Divergence

$$\nabla \cdot \mathbf{v} = \frac{1}{|\det G|^{\frac{1}{2}}} \frac{\partial}{\partial X^j} \left(|\det G|^{\frac{1}{2}} U^j \right)$$

- Gradient

$$(\nabla f)^i = G^{ij} \frac{\partial f}{\partial X^j}$$

- Covariant derivative

$$(\nabla_{\mathbf{u}} \mathbf{v})^i = U^j \frac{\partial V^i}{\partial X^j} + \Gamma^i_{jk} U^j V^k$$

Covariant formulation: 3D Case

- For the 3D spherical case the procedure is the same
- if (λ, θ, r) are geographical longitude, latitude and distance to earth's centre and (X, Y, Z) the model coordinates then

$$
\begin{aligned}
\lambda &= X \\
\theta &= Y \\
r &= \psi(X, Y, Z) = a + H_T(1 - Z) + H_B(X, Y) \sinh \gamma Z / \sinh \gamma
\end{aligned}
$$

- Metric tensor in (X, Y, Z) coordinates

$$
G_{XYZ}(X, Y, Z) = \begin{pmatrix} \psi^2 \cos^2 Y + \psi_X^2 & \psi_X \psi_Y & \psi_X \psi_Z \\ \psi_Y \psi_X & \psi^2 + \psi_Y^2 & \psi_Y \psi_Z \\ \psi_Z \psi_X & \psi_Z \psi_Y & \psi_Z^2 \end{pmatrix}
$$

Covariant formulation: Semi-lagrangian

- For the semi-lagrangian advection parallel transport is used for calculating the difference between contravariant vectors at the departure and arrival points
- The trajectory is calculated using the geodesic equation corresponding to the covariant metric tensor
- In this way the semi-lagrangian scheme has a full covariant formulation. In particular the physical velocity components are not used

Semi-implicit time discretization

- Time stepping scheme is a clasical semi-implicit 3TL
- The semi-implicit formulation follows closely the formulation used with the mass-based vertical coordinate, with the use of a linear model around an isothermal hydrostatic balanced atmosphere at rest
- Nevertheless a flat orography is used in the reference state instead of a constant hydrostatic pressure

Semi-implicit time discretization

- The geometry of the linear model is defined by coordinate transformation which is "flat". In the 2D case it is

$$
\begin{aligned}
x &= X \\
z &= \psi^*(Z) = H_T\,(1 - Z)
\end{aligned}
$$

- From this transformation a linear metric tensor is obtained. Consequently the differential operators of the linear model also change. In the 2D case

$$
G^*_{XZ} = \begin{pmatrix} 1 & 0 \\ 0 & H_T^2 \end{pmatrix}
$$

- The use of $\psi^*(Z)$ in height based vertical coordinate is the counterpart of the use of π^*_S in mass based vertical coordinate

Semi-implicit time discretization

- A 3TL level scheme is represented by the following equation

$$\frac{\mathbf{X}^{\mathbf{n+1}} - \mathbf{X}^{\mathbf{n-1}}}{2\Delta t} = \mathbf{M}(\mathbf{X}^{\mathbf{n}}) - \mathbf{L}(\mathbf{X}^{\mathbf{n}}) + \frac{1-\epsilon}{2}\mathbf{L}(\mathbf{X}^{\mathbf{n-1}}) + \frac{1+\epsilon}{2}\mathbf{L}(\mathbf{X}^{\mathbf{n+1}})$$

- $\mathbf{M}$ is the non linear model and $\mathbf{L}$ the linear model
- ϵ is a decentering factor which increases stability
- $\mathbf{X} = (\mathbf{U}, \mathbf{W}, \mathbf{r}, \mathbf{q})$ is the state vector
- The linear system is solved for $\mathbf{X}^{\mathbf{n+1}}$ in the spectral space

Semi-implicit time discretization

- The following structure equation is obtained, which is similar to the structure equation of the ALADIN model

$$\left(\mathbf{I} - \beta^2 c_*^2 \left(\mathbf{D_X^2} + \mathbf{L_Z}\right) - \beta^4 c_*^2 N_*^2 \,\mathbf{D_X^2}\right) \mathbf{W^{n+1}} = \mathbf{R}_C$$

- where the vertical Laplacian $\mathbf{L_Z}$ and the constants are

$$\mathbf{L_Z} = \frac{1}{H_*^2}\left(\left(\frac{H_*}{H_T}\mathbf{D_Z^2}\right)^2 - \left(\frac{H_*}{H_T}\mathbf{D_Z}\right)\right)$$

$$c_*^2 = \frac{C_p}{C_v} R T^*$$

$$N_*^2 = \frac{g^2}{C_p T^*}$$

$$H_* = \frac{R T^*}{g}$$

$$\beta = (1+\epsilon)\,\Delta t$$

Semi-implicit time discretization

- Contrary to the case of the mass-based vertical coordinate no constraints have to be fulfilled by the vertical operators when deriving the structure equation
- There is not a X term in the divergence due to the use of the contravariant vertical velocity
- The boundary conditions for the contravariant vertical velocity can be included in the spline basis for its representation and therefore they will be automatically fulfilled
- A disadvantage is that the decentering factor must be greater than zero for achieving a similar range of stability (according to the SBH method) than the one obtained with the mass-based coordinate

Semi-implicit time discretization: stability

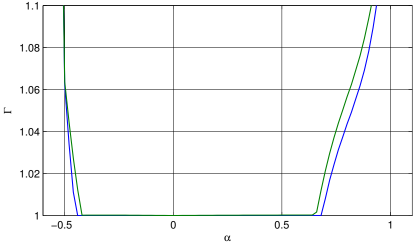
- Study of the stability of the system

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{L}^{\bullet}(\mathbf{X})$$

- $\mathbf{L}^{\bullet}$ is the linear model around an isothermal hydrostatic balanced atmosphere at rest with a reference temperature $T^{\bullet}$ different from T^{*}

Semi-implicit time discretization: stability

- Maximum module of the amplification matrix eigenvalues for different values of the parameter $\alpha = \frac{T^{\bullet}}{T^{*}} - 1$ with $\epsilon = 0.1$

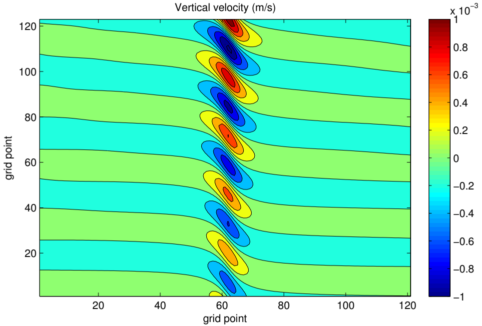


The horizontal wave numbers selected are the corresponding to a horizontal domain of $N_X = 256$ grid points and a horizontal grid spacing of $\Delta x = 2000\,m$. The time step is $\Delta t = 50\,s$. The decentering parameter is set to $\epsilon = 0.1$. The reference temperature of the implicit part is $T^* = 350\,K$. In green, the number of vertical levels are 50 with a regular grid spacing and the top of the atmosphere is placed at 30 km. In blue, the levels are placed at the geopotential height of the 62 levels of the IFS model when using the standard atmosphere

Tests

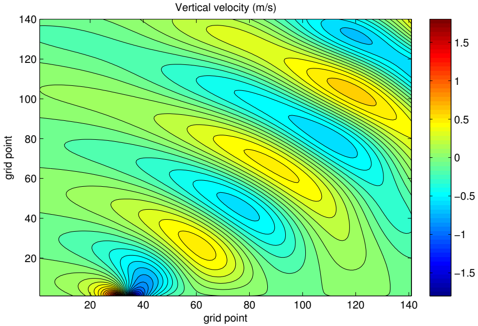
- A set of test cases taken from the literature has been run
  - Linear and non linear hydrostatic and non hydrostatic waves
  - Cold and warm bubbles with diffusion

Tests: Linear hydrostatic wave (Bubnová et al., 1995)



Isothermal atmosphere with $N = 0.02\, s^{-2}$ and $U = 8\, ms^{-1}$ and the bell shaped mountain has $H_0 = 1\, m$ and $a = 16\, km$. The horizontal and vertical resolutions are $\Delta X = 3.2\, km$ and $H_T\, \Delta Z = 100\, m$ with the top of the atmosphere placed at $H_T = 20\, km$. The absorbing layer begins at $10\, km$. The integration is done up to $t^* = tU/a = 120$ with $\Delta t = 90\, s$

Tests: Quasi-linear non hydrostatic wave (Bubnová et al., 1995)



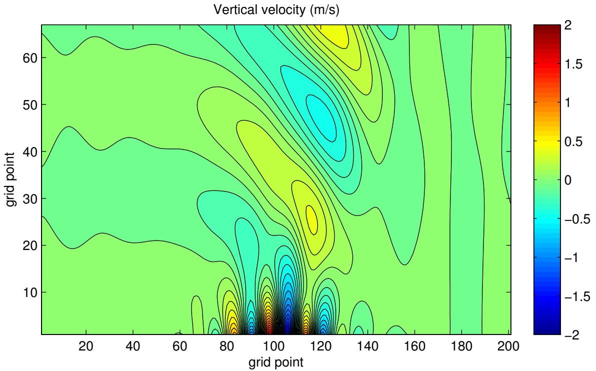
Quasi-linear non hydrostatic wave, with the following changes respect to the previous hydrostatic test:

$U = 15\ ms^{-1}$, $H_0 = 100\ m$ and $a = 500\ m$. The horizontal and vertical resolutions are $\Delta X = 100\ m$ and

$H_T\ \Delta Z = 100\ m$ with the top of the atmosphere placed at $H_T = 20\ km$. The absorbing layer begins at 14 km.

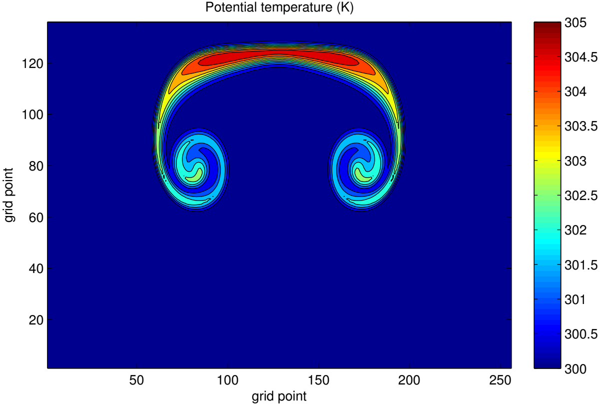
The integration is done up to $t^* = tU/a = 90$ with $\Delta t = 1\ s$

Tests: complex orography (Schär et al., 2002)



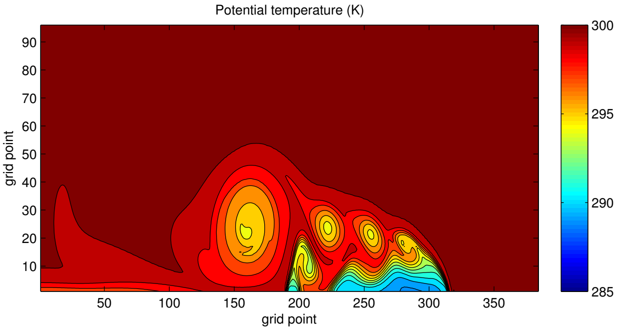
Upstream profile is defined by constant value of the Brunt-Väisälä frequency $N = 0.01\,s^{-1}$ and the horizontal velocity $u = 10\ ms^{-1}$ together with the upstream surface temperature $T = 288\ K$ and pressure $p = 1000\ hPa$. The mountain ridge is a bell shaped structure with superposed small scale features $H_0 = 250\ m$, $a = 5000\ m$ and $b = 4000\ m$. The vertical resolution is 150 m and the horizontal resolutions is 250 m. The number of vertical levels is 130. The time step is $\Delta t = 4\ s$

Tests: warm bubble (Janjic et al., 2001)



The integration domain extended 25.6 km in the horizontal direction and 13.5 km in the vertical. The center of the initial disturbance is in the middle of the domain in the horizontal direction. The horizontal and vertical resolutions are 100 m. The time step is 0.3 s. Cyclic boundary conditions in the horizontal direction are used. The diffusion coefficient is 50 $m^2 s^{-1}$ for both components of the velocity and the temperature

Tests: cold bubble (Straka, 1993)



The integration domain spans 25.6 km in the horizontal direction and 6.5 km in the vertical. The center of the initial disturbance is in the middle of the domain in the horizontal direction, which is the left boundary in the figures. The horizontal and vertical resolutions are 50 m. The time step is 0.15 s. Cyclic boundary conditions in the horizontal direction are used. The diffusion coefficient is 75 $m^2 s^{-1}$ for both components of the velocity and the temperature. Boundary conditions are: vertical velocity is zero at the boundaries and first derivative respect to vertical coordinate of temperature, pressure and horizontal velocity is zero at the boundaries

Conclusions

- 2D non hydrostatic dynamical core previous to a 3D implementation with height based hybrid vertical coordinate and vertical finite elements
- Covariant formulation of all aspects of the model including the semi-lagrangian scheme
- Contravariant vertical velocity as prognostic variable which eases the implementation of the boundary conditions
- Favourable aspects
    - No vertical operator contraints
    - VFE possible with high accuracy in the vertical operators
    - Clean implementation of boundary conditions

Conclusions: Unfavourable aspects

- Stability of the linear model needs a small non-zero decentering factor
- Instability for moderate and high mountain slopes, height/width ratio over 5%
  - The instability seems to be produced by the difference between the linear and non linear terms in the gradient pressure term of the momentum equation
  - This problem seems to be related with the horizontal spectral discretization which needs a horizontally uniform reference state and geometry
  - The range of permitted slopes must be increased before implementing the scheme in a 3D operational model. Some hints: papers about the stability of the ALADIN model and how it was solved.

Thank you for your attention!