Stability Analysis of the Runge-Kutta Time Integration Schemes for the Convection resolving Model COSMO-DE (LMK)

7th SRNWP-Workshop on Non-Hydrostatic Modelling, Bad Orb
05.-07.11.2007

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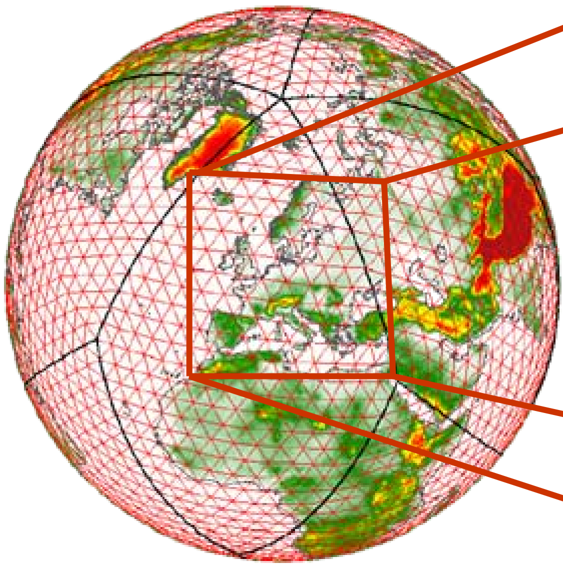
The operational Model Chain of DWD: GME, COSMO-EU and -DE

(since 16. April 2007)

GME

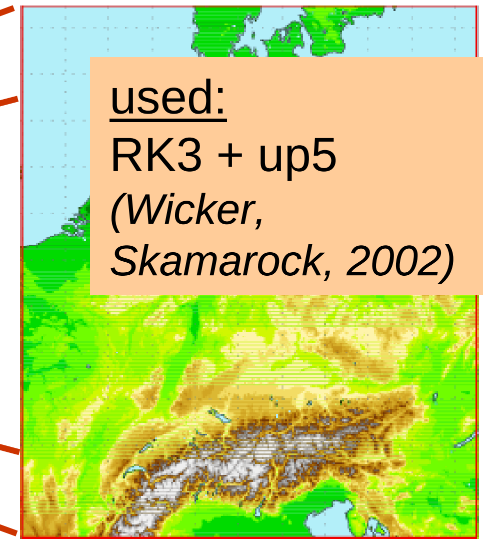


COSMO-EU (LME)

<u>used:</u>
Leapfrog + cd2
(Klemp, Wilhelmson, 1978)
<u>planned:</u>
RK3 + up3

better solution possible?

COSMO-DE (LMK)



<u>used:</u>
RK3 + up5
(Wicker, Skamarock, 2002)

GME

hydrostatic
parameterised convection
$\Delta x \approx 40$ km
368642 * 40 GP
Δt = 133 sec., T = 7 days

COSMO-EU

non-hydrostatic
parameterised convection
$\Delta x = 7$ km
665 * 657 * 40 GP
Δt = 40 sec., T= 78 h

COSMO-DE

non-hydrostatic
resolved convection
$\Delta x = 2.8$ km
421 * 461 * 50 GP
Δt = 25 sec., T = 18 h

Runge-Kutta-Time-Integration

autonomous ODE-system

$$\frac{dq_l}{dt} = f_l(q_1, ..., q_M), \quad l = 1, 2, ..., M$$

explicit *N*-stage Runge-Kutta-method (to integrate from t^n to t^{n+1}):

$$\begin{aligned} q_l^{(0)} &\equiv q_l^n, \\ q_l^{(i)} &= q_l^{(0)} + \Delta t \cdot \sum_{j=1}^{i} \beta_{i+1,j}\, f_l(\mathbf{q}^{(j-1)}), \quad i = 1, 2, ..., N \\ q_l^{n+1} &\equiv q_l^{(N)} \end{aligned}$$

(other representations possible, e.g. ‚substepping-form')

Butcher-Tableau:

$$\begin{array}{c|ccccc} 0 & & & & & \\ \alpha_2 & \beta_{21} & & & & \\ \alpha_3 & \beta_{31} & \beta_{32} & & & \\ \dots & \dots & \dots & & & \\ \alpha_N & \beta_{N1} & \beta_{N2} & \dots & \beta_{N,N-1} & \\ \hline & \beta_{N+1,1} & \beta_{N+1,2} & \dots & \beta_{N+1,N-1} & \beta_{N+1,N} \end{array}$$

$$\alpha_i := \sum_{j=1}^{i-1} \beta_{ij}$$

Consistency condition → at least 1st order accuracy:

$$\sum_{i=1}^{N} \beta_{N+1,i} = 1$$

example:
4 conditions for RK 3rd order:

$$\begin{aligned}
\beta_{41} + \beta_{42} + \beta_{43} &= 1, \\
\beta_{42}\,\beta_{21} + \beta_{43}\,(\beta_{31} + \beta_{32}) &= 1/2, \\
\beta_{42}\,\beta_{21}^2 + \beta_{43}\,(\beta_{31} + \beta_{32})^2 &= 1/3, \\
\beta_{43}\,\beta_{32}\,\beta_{21} &= 1/6,
\end{aligned}$$

stage N	# of β_{ij}	# of cond. for *N* th order RK	# of cond. for *N* th order LC-RK
1	1	1	1
2	3	2	2
3	6	4	3
4	10	8	4
5	15	12(?)	5
...	...	...	...

Standard-Testproblem

$$\frac{d\mathbf{q}}{dt} = \mathbf{Pq}$$

linear, homogeneous ODE-system (*M* equations) with time-independent *M***M* Matrix $\mathbf{P}$

Theorem: for the Standard-Testproblem, a *N*-stage RK-method is of **order *N***, if the N conditions

$$h_{N+1}^{(l)} = \frac{1}{l!}, \qquad l = 1, 2, ..., N$$

hold. Such a scheme is called here **Linear Case-Runge-Kutta (LC-RK) of order *N*.**
($h_N^{(l)}$ are polynomials in the β_{ij} , they can easily be calculated by a recursion formula)

example: the ‚3rd order RK method' used for WRF *(Wicker, Skamarock, 2002)* is a 3-stage, 2nd order RK, but a 3rd order LC-RK

Lemma: all LC-RK methods of order *N* behave similar for the standard testproblem.

Lemma:
All ‚pure' RK methods of order *N* (i.e. *N*-stage, *N*th order) are a subset of the LC-RK methods of order *N*

→ all ‚pure' RK methods of the same order have the same stability properties for the standard testproblem!

Linear advection equation (1-dim.)

$$\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0$$

Spatial discretisations of the advection operator (order 1 ... 6)

$$f_j^{(1)}(q) := -u \frac{q_j - q_{j-1}}{\Delta x}$$

upwind 1st order

$$f_j^{(2)}(q) := -u \frac{q_{j+1} - q_{j-1}}{2\,\Delta x}$$

centered diff. 2nd order

$$f_j^{(3)}(q) := -u \frac{2q_{j+1} + 3q_j - 6q_{j-1} + q_{j-2}}{6\,\Delta x}$$

$$f_j^{(4)}(q) := -u \frac{-(q_{j+2} - q_{j-2}) + 8(q_{j+1} - q_{j-1})}{12\,\Delta x}$$

$$f_j^{(5)}(q) := -u \frac{-3q_{j+2} + 30q_{j+1} + 20q_j - 60q_{j-1} + 15q_{j-2} - 2q_{j-3}}{60\,\Delta x}$$

$$f_j^{(6)}(q) := -u \frac{(q_{j+3} - q_{j-3}) - 9(q_{j+2} - q_{j-2}) + 45(q_{j+1} - q_{j-1})}{60\,\Delta x}$$

from the Theorem and Lemma above:
all *N*th order (LC-) Runge-Kutta-schemes have the same linear stability properties for *u*=const. !

Stability limit for the ‚effective Courant-number' for advection schemes

$C_{eff} := C / s$,   s = stage of RK-scheme

	up1	cd2	up3	cd4	up5	cd6
Euler	1	0	0	0	0	0
LC-RK2	0.5	0	0.437	0	0	0
LC-RK3	0.419	0.577	**0.542**	0.421	**0.478**	0.364
LC-RK4	0.348	**0.707**	0.436	**0.515**	0.433	0.446
LC-RK5	0.322	0	0.391	0	0.329	0
LC-RK6	0.296	0	0.385	0	0.311	0
LC-RK7	0.282	0.252	0.369	0.184	0.323	0.159

Baldauf (2007), submitted to JCP

time splitting (or **sub-cycling**):
distinguish between slow (adv.) and fast (sound, buoyancy) processes
→ Matrix **P** in the standard test problem becomes time-dependent!
→ general stability statements from above not applicable ☹
nevertheless LC-RK is not only of academic interest ...

Stability analysis of a 2D (horizontal + vertical) system with Sound + Buoyancy + Advection (+ Smoothing, Filtering)

Restrictions:

- no boundaries (wave expansion in ∞ extended medium)
- base state: p_0=const, T_0=const   (for the coefficients)
  (but stratification $dT_0/dz \neq 0$ possible)
  → application to structures with a vertical extension of about 2-3 km
- no orography (i.e. no metric terms)
- only horizontal advection

Linearised, 2-dim. Sound-Buoyancy-Advection-System

$$\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + Q_x$$

$$\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \left(\frac{p'}{p_0} - \frac{T'}{T_0}\right) g + Q_z$$

$$\frac{\partial p'}{\partial t} + U_0 \frac{\partial p'}{\partial x} = -\frac{c_p}{c_v} p_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \rho_0 g w$$

$$\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} = -\frac{R}{c_v} T_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \frac{\partial T_0}{\partial z} w$$

Tend. Adv. Sound Buoyancy

- primitive variables (*p*‘*T*‘-Dynamics)

$$u, \quad w, \quad p = p_0 + p', \quad T = T_0 + T'$$

- Q_x, Q_z e.g. a divergence damping

$$Q_x = \alpha_{div} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right), \quad Q_z = \alpha_{div} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

Buoyancy terms:

$$\frac{w^{n+1} - w^n}{\Delta t} = g\left(\beta_T^b \frac{T'^{n+1}}{T_0} + (1-\beta_T^b)\frac{T'^n}{T_0} - \beta_p^b \frac{p'^{n+1}}{p_0} - (1-\beta_p^b)\frac{p'^n}{p_0}\right)$$

$$\frac{p'^{n+1} - p'^n}{\Delta t} = \rho_0 g(\beta_3^b\, w^{n+1} + (1-\beta_3^b)\, w^n)$$

$$\frac{T'^{n+1} - T'^n}{\Delta t} = -\frac{\partial T_0}{\partial z}(\beta_4^b\, w^{n+1} + (1-\beta_4^b)\, w^n)$$

acoustic cut-off frequency $\omega_a := \sqrt{N^2 + \frac{g^2}{c_S^2}}$; $C_{buoy} := \omega_a \Delta t$

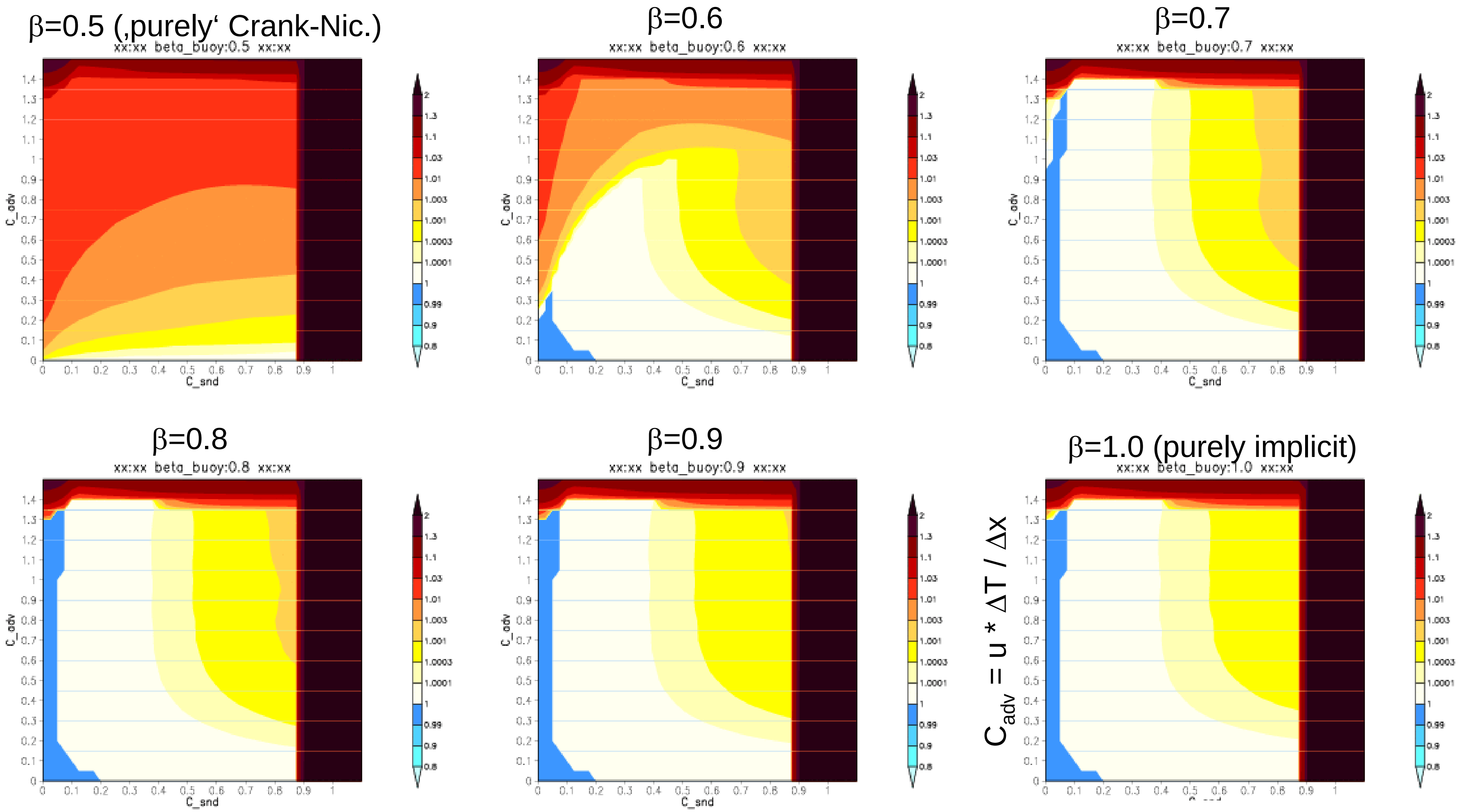
$C_\beta = \frac{1}{T_0}\frac{\partial T_0}{\partial z}\frac{c_S^2}{g} \approx -0.24$ (standard atmosphere)

fully explicit	unstable	-
forward-backward	stable for $C_{buoy} < 2$	neutral
Crank-Nicholson $\beta = 1/2$	uncond. stable	neutral
Crank-Nicholson $\beta > 1/2$	uncond. stable	damping
implicit	uncond. stable	damping

Commutation with other numerical operators:

$Q_{buoy} \cdot Q_{sound} \neq Q_{sound} \cdot Q_{buoy}$, $\qquad Q_{buoy} \cdot Q_{div} \neq Q_{div} \cdot Q_{buoy}$

Choice of CN-parameters for buoyancy in the *p'T'*-Dynamic of COSMO-DE



$C_{snd} = c_s * \Delta t / \Delta x$

→ **choose β=0.7 as the best value**

Comparison between 3-stage RK-schemes

Motivation: a model crash during the storm ‚Kyrill' (‚18. Jan. 2007')

'Simplest'-LC-RK3
(3-stage, 2nd order)
(Wicker, Skamarock, 2002)

0			
1/3	1/3		
1/2	0	1/2	
	0	0	1

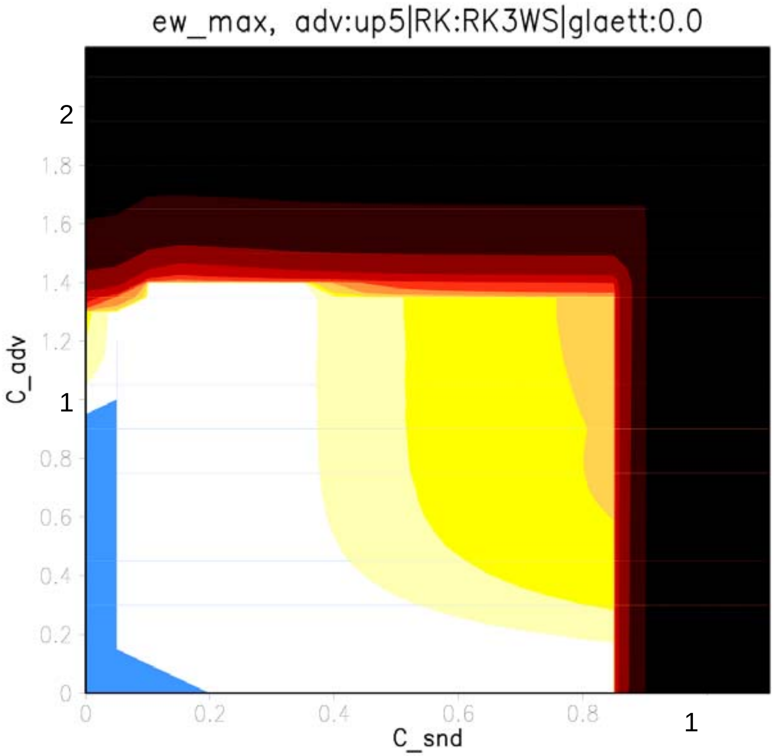
TVD-RK3
(3-stage, 3rd order)
(Fehlberg, 1970;
Shu, Osher, 1988;
Hundsdorfer et al., 1995)

0			
1	1		
1/2	1/4	1/4	
	1/6	1/6	2/3

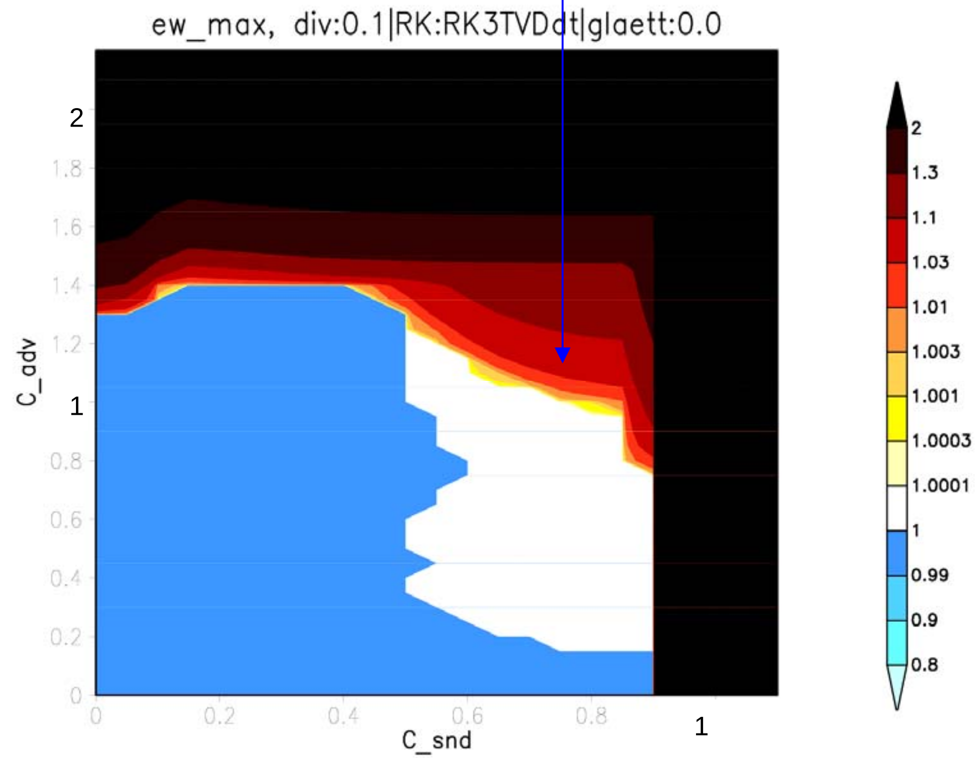
Configuration of the following tests:

- Buoyancy: ~ standard atmosphere, off-centering β=0.7
- Sound: off-centering β=0.7
- aspect ratio: $\Delta x/\Delta z$=10
- with divergence damping (C_{div} = 0.1)

S-LC-RK3
(3-stage, 2nd order)
+ up5
(Wicker, Skamarock, 2002)



TVD-RK3 *(Shu, Osher, 1988)*
+ up5
different small timesteps →
$\Delta T / \Delta t = 6$ possible



'Simplest'-LC-RK4
(4-stage, 2nd order)

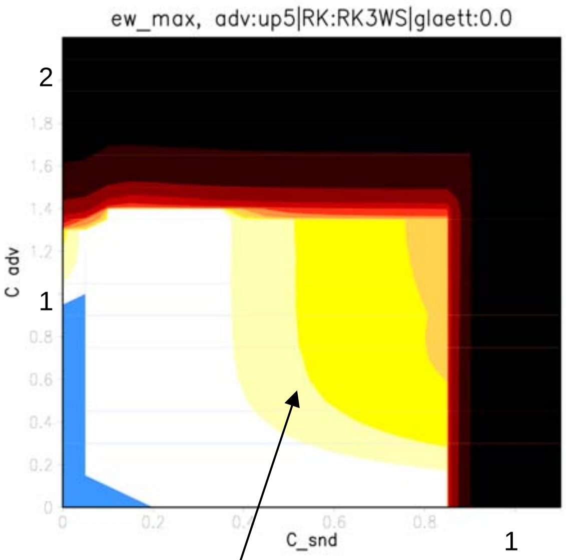
0				
¼	¼			
1/3	0	1/3		
½	0	0	½	
	0	0	0	1

Are 4-stage RK-schemes competitive?

S-LC-RK3 (3-stage, 2nd order) **+ up5**

$\Delta T/\Delta t = 6$

no smoothing

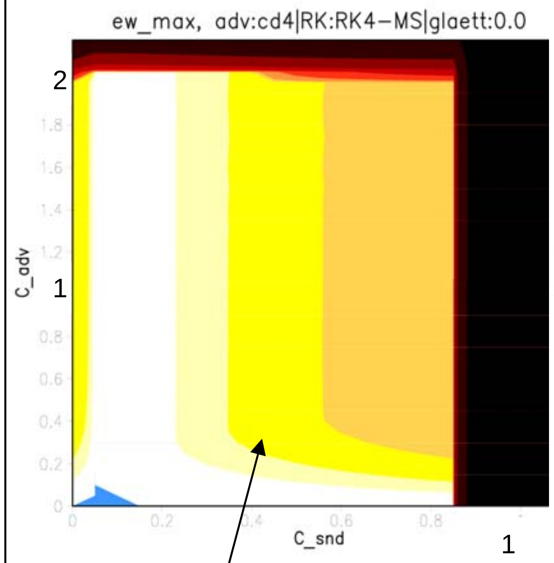


instab. by long waves

S-LC-RK4 (4-stage, 2nd order) **+ cd4**

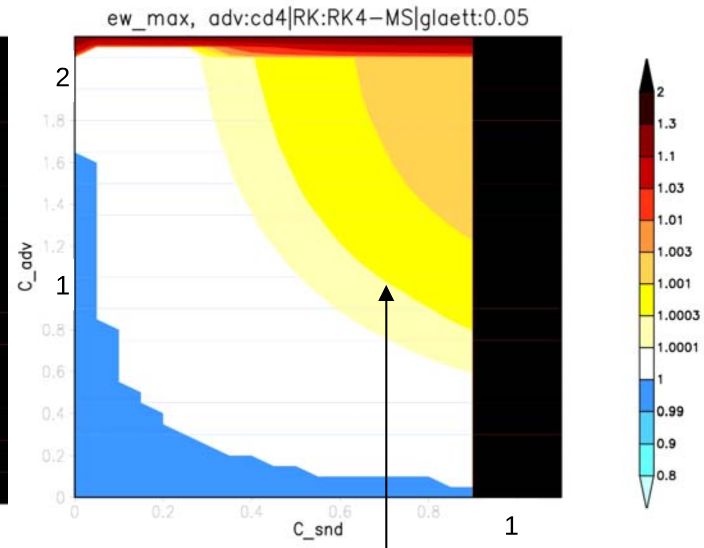
$\Delta T/\Delta t = 12$

no smoothing



instab. by short waves

\+ 4th order diffusion



instab. by long waves

Are 4-stage RK-schemes competitive?

‘Simplest'-LC-RK4
(4-stage, 2nd order)

0				
¼	¼			
1/3	0	1/3		
½	0	0	½	
	0	0	0	1

'classical' RK4
(4-stage, 4th order)
(Numerical recipies)

0				
½	½			
½	0	½		
1	0	0	1	
	1/6	1/3	1/3	1/6

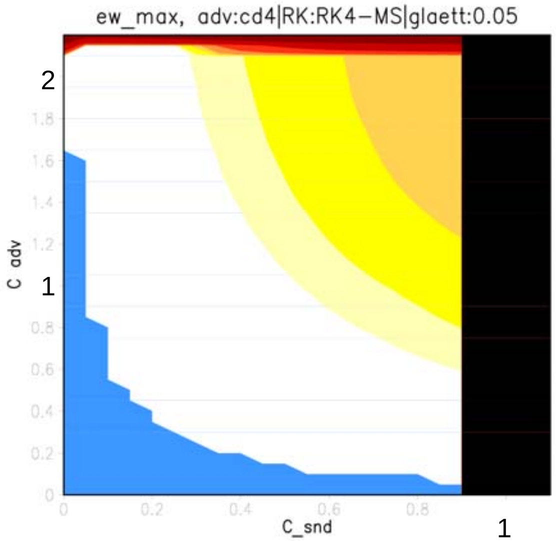
(negative coeff. arises if written in 'substepping-form')

RK-SSP(4,3)
(4-stage, 3rd order)
(Ruuth, Spiteri, 2004)

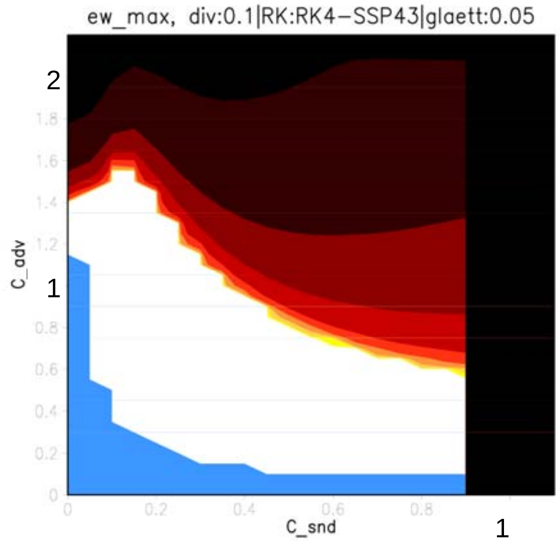
0				
1/2	1/2			
1	1/2	1/2		
1/2	1/6	1/6	1/6	
	1/6	1/6	1/6	1/2

Other RK4 + cd4 + add. smoothing ...

S-LC-RK4
(4-stage, 2nd order)

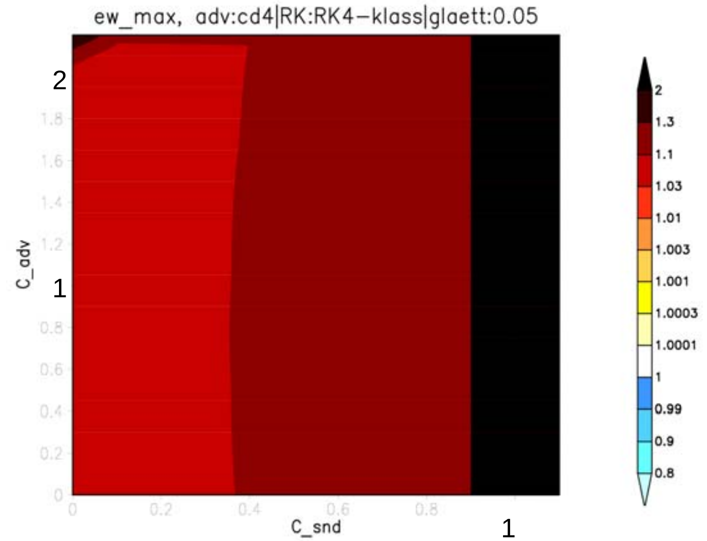


SSP(4,3)-RK
4-stage, 3rd order
(Ruuth, Spiteri, 2004)
<u>is not a LC-RK-scheme!</u>



strong stability preserving schemes do not automatically work together with timesplitting

'classical' RK4
(4-stage, 4th order)



completely unstable (!?) due to negative coefficients in the ‚substepping-form'?

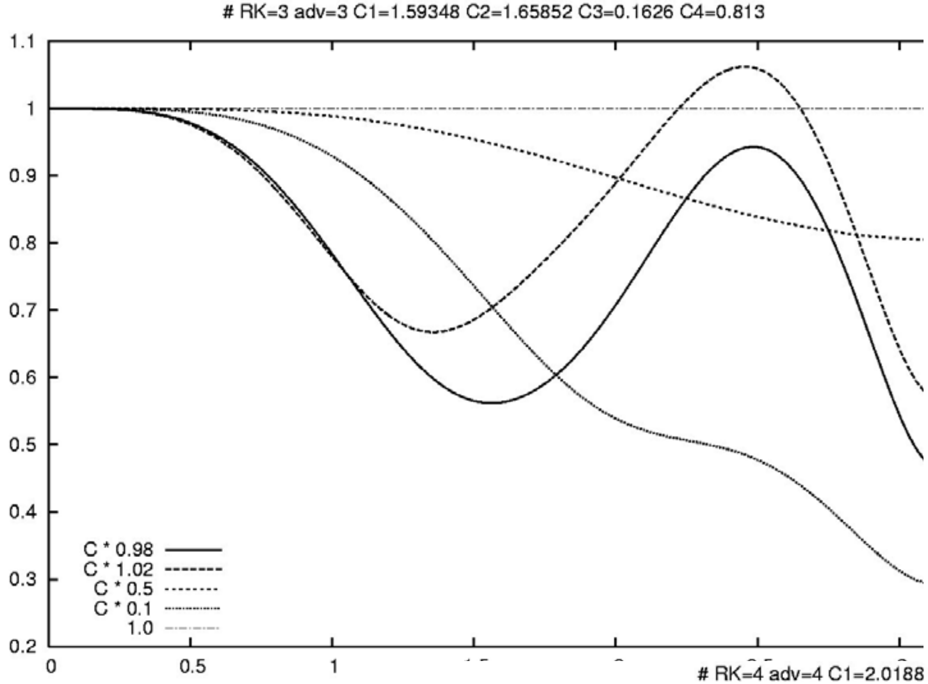
Summary

- rather general RK-theory for the ‚Standard-Testproblem' developed
- application to the stability problem of 1D advection equation (partly analytical results for critical Courant numbers)
- linear stability of 2D Sound-Buoyancy-Advection-system
- determination of Crank-Nicolson-coeff. for sound and buoyancy discretization
- TVD-RK3 reduces spurious instabilities but also reduces the possible range for C_{adv}
- RK4 + cd4 + add. smoothing could be a more efficient alternative to RK3 + up5; with better advection properties than RK3 + up3
- careful choice of the 4-stage RK is needed. Is a better RK4 possible for time-splitting?

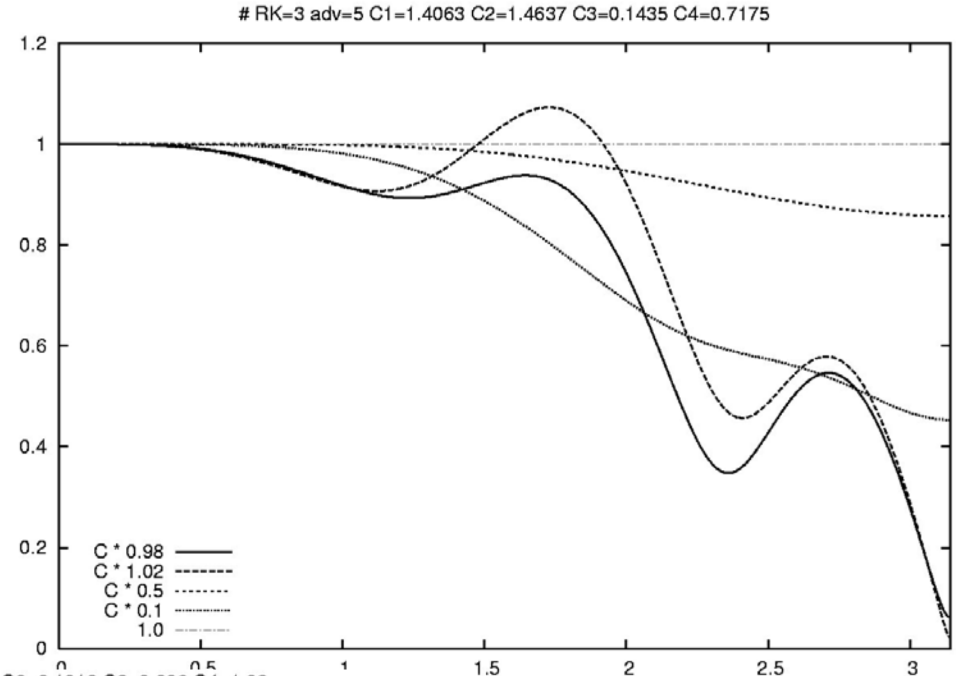
Stable Courant-numbers for advection schemes

	up1	cd2	up3	cd4	up5	cd6
Leapfrog	0	1	0	0.72	0	0.62
LC-RK1 (Euler)	1	0	0	0	0	0
LC-RK2	1	0	0.874	0	0	0
LC-RK3	1.256	1.732	1.626	1.262	1.435	1.092
LC-RK4	1.393	2.828	1.745	2.061	1.732	1.783
LC-RK5	1.609	0	1.953	0	1.644	0
LC-RK6	1.777	0	2.310	0	1.867	0
LC-RK7	1.977	1.764	2.586	1.286	2.261	1.113

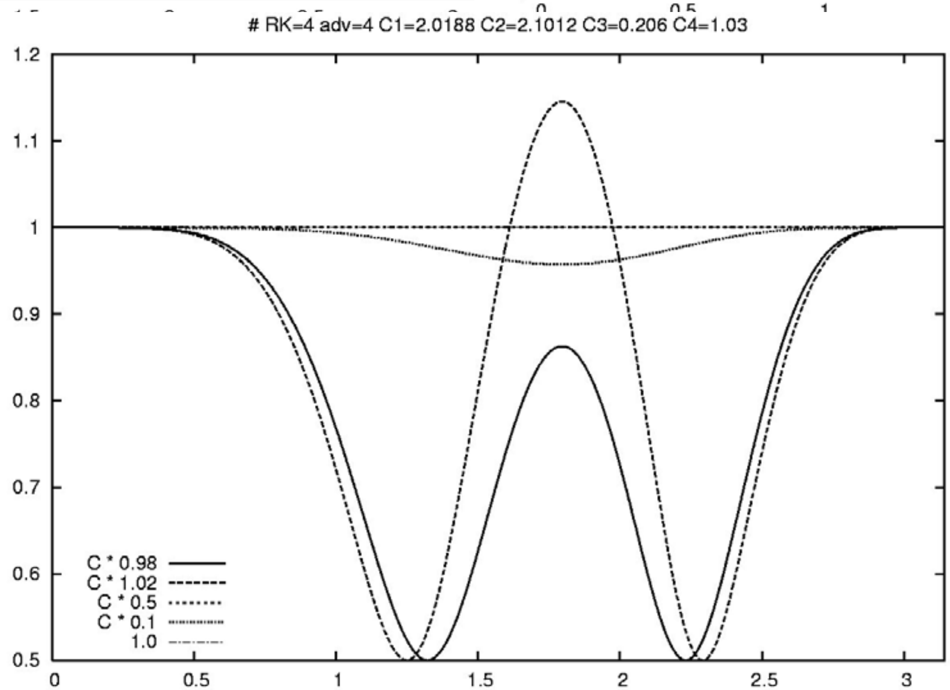
Wicker, Skamarock, 2002: Leapfrog, RK2, RK3



RK=3 adv=3 C1=1.59348 C2=1.65852 C3=0.1626 C4=0.813
C * 0.98
C * 1.02
C * 0.5
C * 0.1
1.0



RK=3 adv=5 C1=1.4063 C2=1.4637 C3=0.1435 C4=0.7175
C * 0.98
C * 1.02
C * 0.5
C * 0.1
1.0



RK=4 adv=4 C1=2.0188 C2=2.1012 C3=0.206 C4=1.03
C * 0.98
C * 1.02
C * 0.5
C * 0.1
1.0

Stabilitätsanalyse Tool

von-Neumann-Analyse für je eine Komb. von c_s, U, dT_0/dz, ... :

- bestimme die Stabilitätsmatrix Q (4*4-Matrizen)
- berechne deren Eigenwerte (Verwendung von LAPACK-EW-Routinen)
- Suche den maximalen Eigenwert durch Abscannen im Bereich $k_x \Delta x = -\pi ... + \pi$, $k_z \Delta z = -\pi ... + \pi$ ab.

‘Verifikation’:

- analytisch bekannte Stabilitätsgrenzen (Advektion, Schall, Divergenzdämpfung) werden korrekt berechnet
- physikalische Schichtungsinstabilität (d.h. $N^2<0$) wird wiedergegeben (→ auch Kombination ‘Auftrieb + Schall’ funktioniert)

Sound

$$\frac{u^{n+1}-u^n}{\Delta t} = -\frac{1}{\rho_0}\left(\beta_1^s\,\delta_x p'^{n+1} + (1-\beta_1^s)\,\delta_x p'^n\right)$$

$$\frac{w^{n+1}-w^n}{\Delta t} = -\frac{1}{\rho_0}\left(\beta_2^s\,\delta_z p'^{n+1} + (1-\beta_2^s)\,\delta_z p'^n\right)$$

$$\frac{p^{n+1}-p^n}{\Delta t} = -\frac{c_p}{c_v}p_0\left(\beta_3^s\,\delta_x u^{n+1} + (1-\beta_3^s)\,\delta_x u^n + \beta_4^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^n\right)$$

$$\frac{T^{n+1}-T^n}{\Delta t} = -\frac{R}{c_v}T_0\left(\beta_5^s\,\delta_x u^{n+1} + (1-\beta_5^s)\,\delta_x u^n + \beta_6^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^n\right)$$

- temporal discret.: 'generalized' Crank-Nicholson
  β=1: implicit, β=0: explicit
- spatial discret.: centered diff.

Courant-numbers: $C_{snd,x} = c_s\frac{\Delta t}{\Delta x}$, $C_{snd,z} = c_s\frac{\Delta t}{\Delta z}$ $c_s^2 = \frac{c_p}{c_v}\frac{p_0}{\rho_0}$

fully explicit	uncond. unstable	-	
forward-backward (Mesinger, 1977), unstaggered grid	stable for $C_x^2+C_z^2<2$	neutral	4 dx, 4dz
forward-backward, staggered grid	stable for $C_x^2+C_z^2<1$	neutral	2 dx, 2dz
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}$=1/2)	stable for $C_x<1$	neutral	2 dx
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}$>1/2)	stable for $C_x<1$	damping	2 dx

Divergence damping

$$\frac{\partial \mathbf{v}}{\partial t} + \ldots = \ldots + \alpha_{div} \nabla (\operatorname{div} \mathbf{v})$$

$\Rightarrow$ Diffusion of Divergence

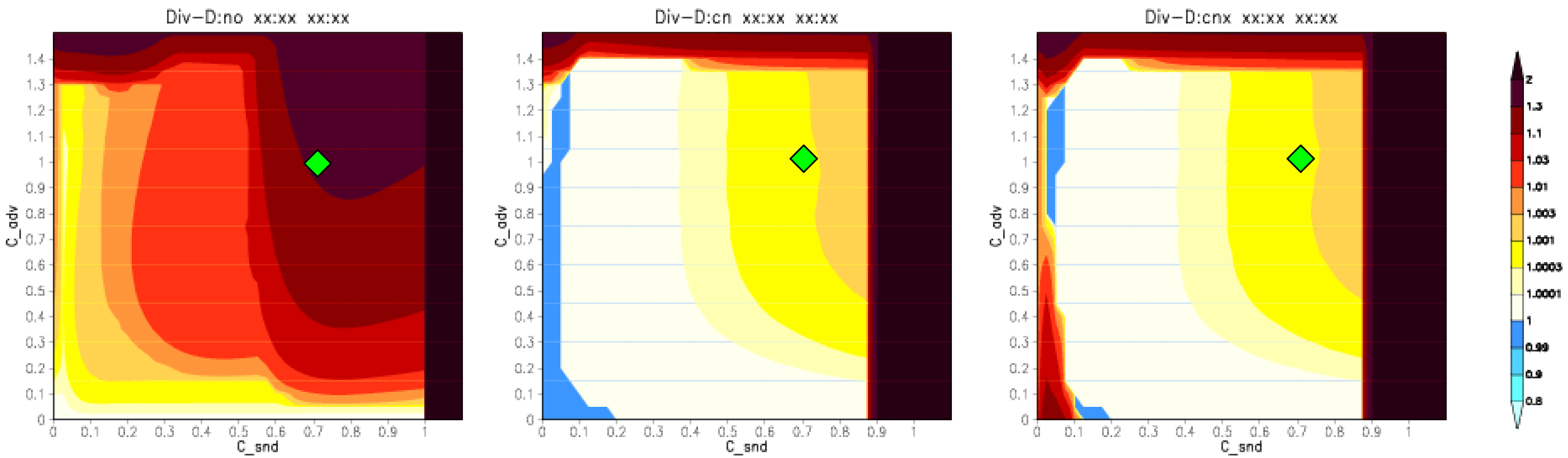
Courant-numbers: $C_{div,x} := \alpha_{div} \frac{\Delta t}{\Delta x^2}, \quad C_{div,z} := \alpha_{div} \frac{\Delta t}{\Delta z^2}$

1D, explicit	stable for $C_{div,x} < \frac{1}{2}$
2D, explicit (staggered grid)	stable for $C_{div,x} + C_{div,z} < \frac{1}{2}$
2D, implicit	unconditionally stable
2D, vertically implicit	stable for $C_{div,x} < \frac{1}{2}$, $C_{div,z}$ arbitr.

Wicker, Skamarock (2002): $\alpha_{div} \approx 0.1\, c_s^2\, \Delta t$ $\Rightarrow$ for LMK: $\alpha_{div} \approx 50000 m^2/s$

↑
xkd

Need of a divergence damping:

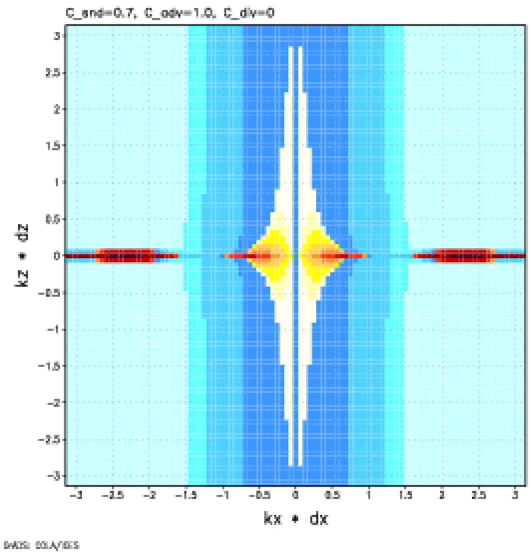


without divergence damping

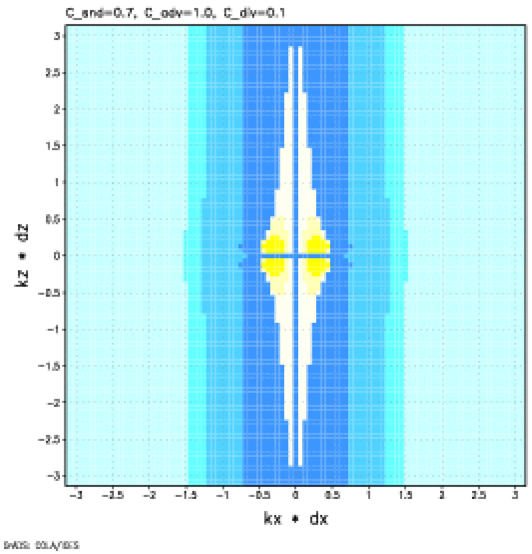
with 3D-(isotropic)-divergence damping

with quasi-3D (non-isotropic) divergence damping

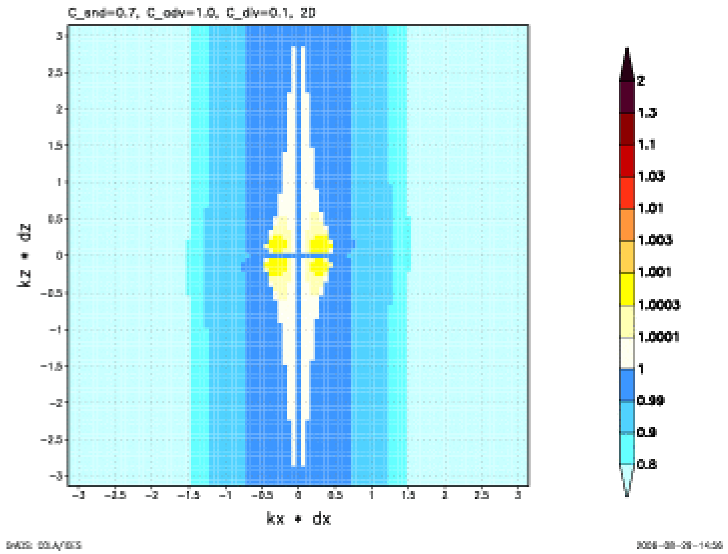
Stability of single waves in C_{adv}=1, C_{snd}=0.7



without divergence damping



with 3D-(isotropic)-
divergence damping



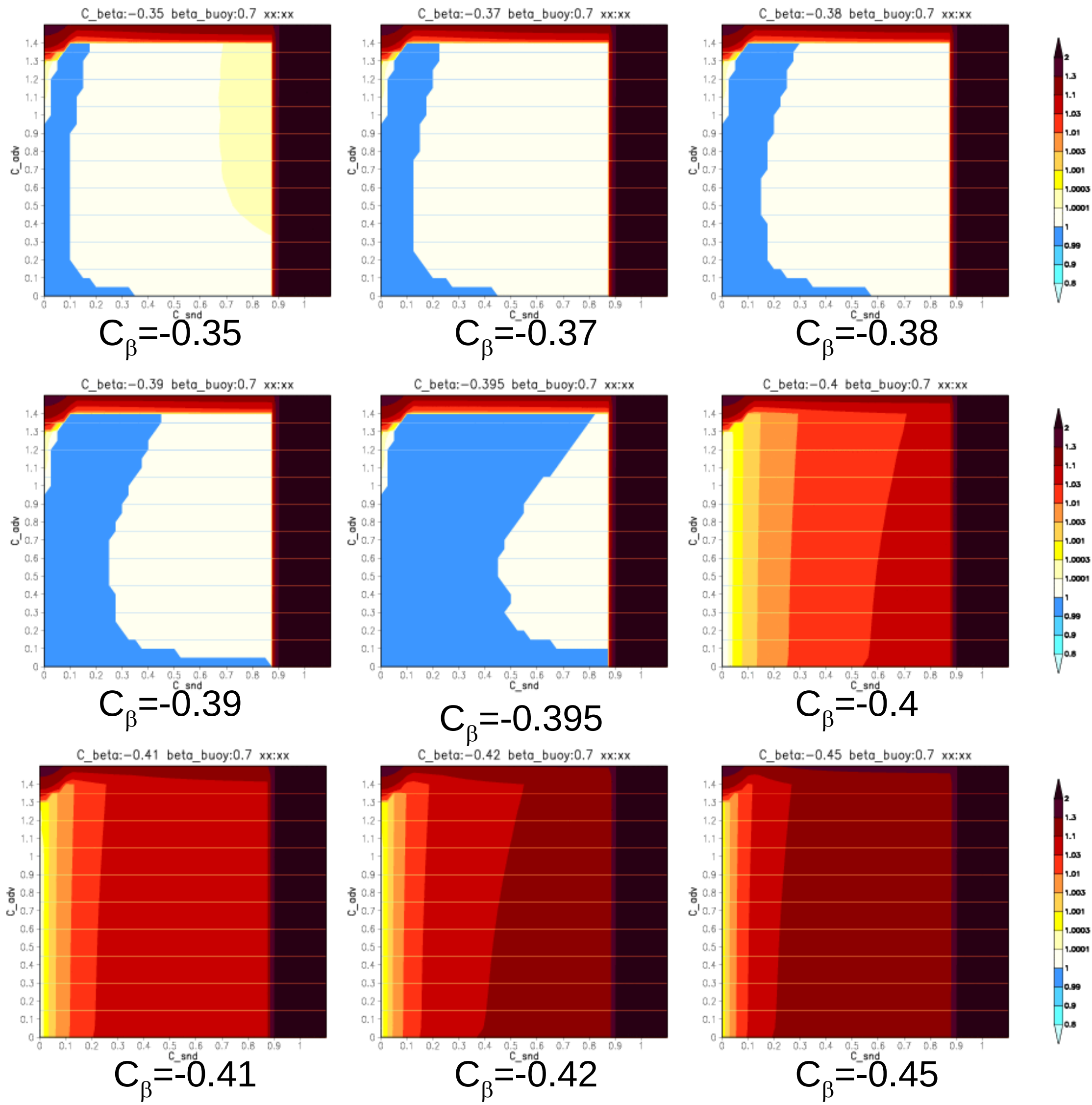
with quasi-3D (non-isotropic)
divergence damping

‚Verification‘ of the stability analysis tool:
dependency from stratification

$$C_\beta := \frac{1}{T_0}\frac{dT_0}{dz}\frac{c_s^2}{g}$$

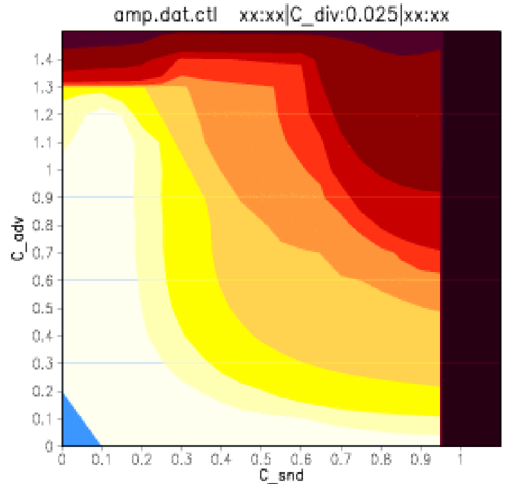
critical value:
C_β=-0.399 ←→ N=0

→ Tool works for ‚buoyancy + sound‘

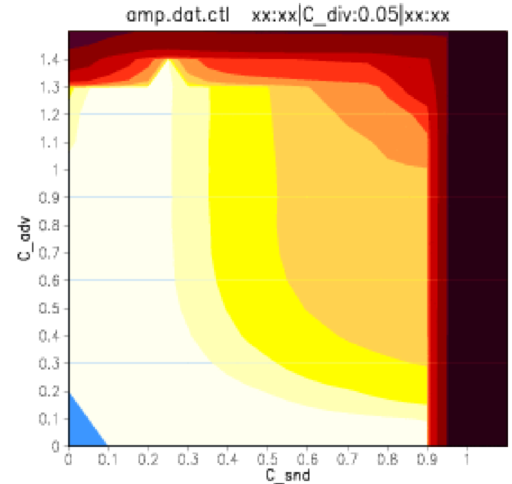


Influence of C_{div}

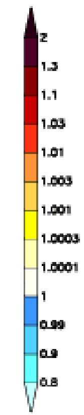
$$C_{div} = \frac{xkd * (c_s * \Delta t / \Delta x)^2}{\sim 0.35}$$

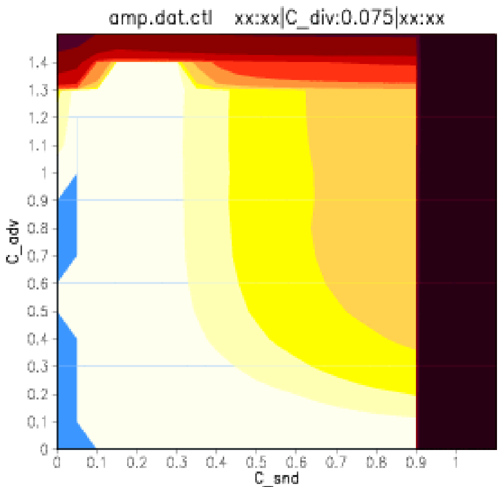


C_{div}=0.025

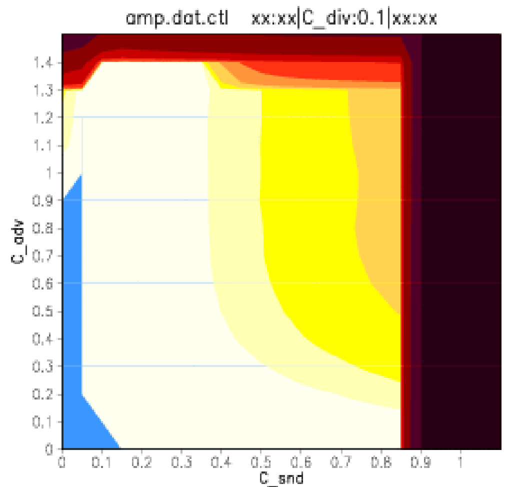


C_{div}=0.05

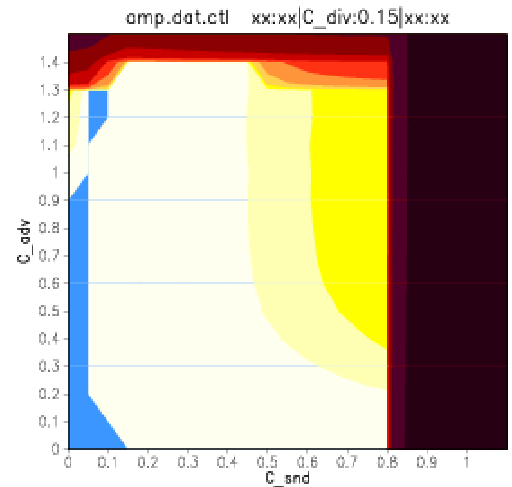




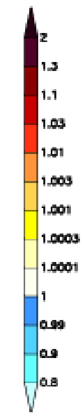
C_{div}=0.075



C_{div}=0.1



C_{div}=0.15

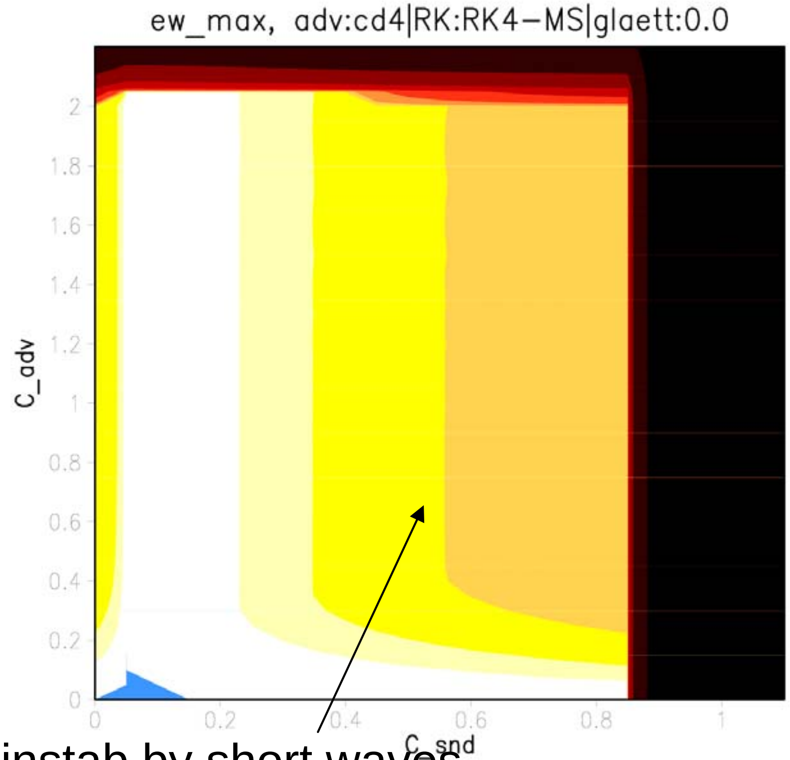


Configuration of following tests:

- Buoyancy: ~ standard atmosphere, off-centering β=0.7
- Sound: off-centering β=0.7
- aspect ratio: $\Delta x / \Delta z$=10
- with div.damping (C_{div} = 0.1)

4-stage, 2nd order RK (MS-LC-RK4) + cd4

no smoothing



instab by short waves

+ 4th order diffusion